Nonlinear variational model for image segmentation problem

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EXTENDED ABSTRACT

In this work, we use the $q(\cdot)$-Laplace energy to speckle-noise removal, medical ultrasound imaging segmentation and important features detection based on the topological gradient method. The approach consists in minimizing a $q(\cdot)$-Laplace energy with a variable exponent function $q(\cdot)$. Then, we consider an adaptive choice of $q(\cdot)$ based on the topological sensitivity approach. We also consider the watershed technique for the segmentation problem. We present several numerical examples to show the performance of the proposed method.

Keywords: $q(x)$-Laplace operator, image segmentation, topological gradient.

1 Introduction

Images processing such as image denoising, edge detection and image segmentation, etc, plays an important role in various fields. The objective of image denoising ultrasound image is to reconstruct an image $u > 0$ from a noisy one

$$f = u + v\sqrt{u}, \tag{1}$$

where $v > 0$ follows the Rayleigh-distribution. Segmenting an image is to separate objects of interest from each others or from the background, or to find boundaries of such objects. Model (1) represents the degradation of an image corrupted by a multiplicative noise, mainly speckle noise, usually present in medical ultrasound imaging [3, 5].

In segmentation, the quality of the result strongly depends on the quality of the initial noisy image. As we aim to get better and meaningful segmentation, we will improve the quality of the reconstructed image by considering the following restoration model:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{q(x)-2} u) + \alpha \frac{u^2-f^2}{u^2} = 0, & \text{in } \Omega, \\ \partial_n u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $q : \Omega \to]1, 2]$ is measurable function called exponent and the parameter α is a positive weight. For more details about the exponent functions and Orlicz spaces, we refer the reader to [2, 6].

The variable exponent $1 < q(x) \leq 2$ is chosen so that to slow diffusion near edges in order to highlight them, and to enhance diffusion in smooth regions. A classical idea of choosing the values of the exponent q is to make an adaptive procedure as follows: first, we consider the topological gradient method with $q(x) = 2$ to identify the edges in order to preserve them. Second, we perform a local selection of the exponent $1 < q(x) \leq 2$ with the help of the map furnished by the topological gradient.

2 Well-posedness

Let Ω be a bounded and Lipschitz open subset in $\mathbb{R}^2$. In the following theorem, we establish the well-posedness of equation (2).

Theorem 2.1. *Let $f \in W^{1,q(x)}(\Omega) \cap L^2(\Omega)$ with $\inf_\Omega f > 0$. Then, equation (2) admits a unique solution u in $W(\Omega) := \{u > 0;\, u \in W^{1,q(x)}(\Omega)\}$ satisfying the maximum principal*

$$\inf_\Omega f \leq u \leq \sup_\Omega f.$$

Proof. First, we note that (2) is the Euler-Lagrange equation of the following minimization problem

$$\min_{\{u \in W^{1,q(x)}\}} \left\{ E(u) := \int_\Omega |\nabla u|^{q(x)} \, \mathrm{d}x + \alpha \int_\Omega \frac{(f-u)^2}{u} \, \mathrm{d}x \right\}. \tag{3}$$

By the classical compactness, the semi-continuity and the convexity arguments of the energy $E(\cdot)$, it is easy to verify that (3) has a unique minimizer, which is equivalently the unique weak solution of (2). □

3 Edges-detection and preservation

The big challenge in image restoration and segmentation problems is how to accurately detect features such as arteries, filaments, internal organ, etc. To meet with this challenge, we employ here the topological gradient method which was widely used in edge detection [3, 4]. We start by inserting a small crack $\sigma_\varepsilon := \{x_0 + \varepsilon\sigma(n)\}$ in the domain Ω, where $x_0 \in \Omega$, $\varepsilon > 0$, $\sigma(n)$ is a straight crack, and n is a unit vector normal to the crack and we minimize the cost function

$$j(\Omega_\varepsilon) := J(u_\varepsilon) = \int_{\Omega\setminus\sigma_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}x,$$

where u_ε is the unique solution of the following equation defined on the *perturbed domain* $\Omega_\varepsilon \stackrel{\text{def}}{=} \Omega \setminus \ \overline{\sigma}_\varepsilon$:

$$\begin{cases} -\Delta u_\varepsilon + \alpha\frac{u_\varepsilon^2 - f^2}{u_\varepsilon^2} = 0, & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \cup \sigma_\varepsilon, \end{cases} \tag{4}$$

After that, we measure the impact of such a modification of the domain on this cost function by computing the following asymptotic expansion as ε goes to zero

$$J(u_\varepsilon) - J(u_0) = \rho^2 G(x_0, n) + o(\rho^2),$$

where $G(x_0, n)$ is the topological gradient given by [3, 4]:

$$G(x_0, n) = -\pi \nabla u_0(x_0) \cdot n \nabla v_0(x_0) n, \tag{5}$$

and v is the solution of the adjoint problem

$$\begin{cases} -\Delta v + \alpha\frac{2f^2}{u^3} v = \Delta u, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

4 Numerical computation

4.1 Algorithm

The steps of the restoration algorithm are the following:

Algorithm 1 Main algorithm

Given f and α.

1. For $q(\cdot) \equiv 2$, compute u and v which solve equations (4) and (6), respectively.
2. Compute the topological gradient $G(x_0, n)$ for each point $x_0 \in \Omega$.
3. Update $q(\cdot)$ as function on $G(x_0, n)$ to obtain a new exponent and solve (2).

In order to update to exponent $q(\cdot)$, we use the following formula

$$q_a(x) = 1 + \exp(-\mu |G(x, n)|), \ \forall x \in \Omega,$$

where $\mu > 0$ is a constant. However, for such a choice of $q(\cdot)$, equation (2) is strongly nonlinear. For that, we introduce an artificial time variable and we descritize the evolution equation by the finite difference scheme

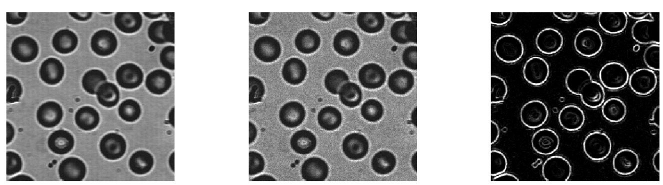
$$\frac{u_{k+1} - u_k}{\tau} = \operatorname{div}(\kappa_\delta(|\nabla u_k|) u_{k+1}) - \alpha \frac{u_k^2 - f^2}{u_k^2},$$

where τ be the time-step, $\kappa_\delta(s) = (\delta^2 + s^2)^{\frac{q(x)}{2}}$ and δ is small constant to avoid the singularity.

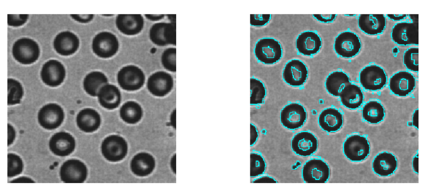
After the restoration of the initial image, we can now proceed for the segmentation problem. In broad terms, image segmentation consists in dividing an image domain into disjoint regions according to a characterization of the image within or in-between the regions. We consider the watershed method which is quite simple: if we consider the image as a topographic relief, where the height of each point is directly related to its gray-level value, and consider rain gradually falling on the terrain, then the watersheds are the lines that separate the “lakes”, called catchment basins that form. Generally, it is computed on the gradient of the original image, so that the catchment basin boundaries are located at high gradient points [1].

4.2 Numerical results

We present here some numerical results in order to show the efficiency of our method. In Figure 1(c), we observe that the blood vessels are well detected by the topological gradient method. The main difference between the noisy image (Figure 1(b)) and restored one (Figure 1(b)) is compared quantitatively by using the SNR and SSIM indicators. The segmented image is presented in Figure 1(e). We remark that all the regions of the image are well identified. The remark hold true for the synthetic image presented in Figure 2.

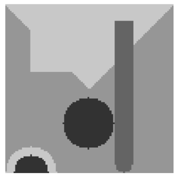


(a) Original image. (b) Noisy image (SSIM =0.69, SNR =20.05). (c) Edges of the image.

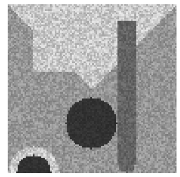


(d) Restored image (SSIM =0.89, SNR =24.86). (e) Segmented image.

Figure 1: From left to right and top to bottom: Original image, noisy image, detected edges, restored image and segmented one.

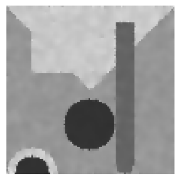


(a) Original image.

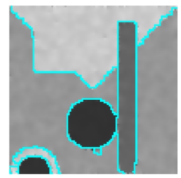


(b) Noisy image (SNR=20.02dB, SSIM=0.41).

(c) Edges of the image.



(d) Restored image (SNR= 30.42, SSIM=0.93).



(e) Segmented image.

Figure 2: From left to right and top to bottom: Original image, noisy image, detected edges, restored image and segmented one.

References

[1] S. Beucher and C. Lantuéjoul, *Use of watersheds in contour detection*, Proc. Int. Worksh. Image Process., Real-Time Edge Motion Detection/Estimation, pp. 17–21, 1979.

[2] X. Fan and D. Zhao, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., vol. 263, pp. 424–446, 2001.

[3] H. Houichet, M. Moakher, and B. Rjaibi, *A topological sensitivity approach for noise removal and edge detection in ultrasound images*, submitted to Applied Mathematical Modeling, p. 23, 2016.

[4] L. Jaafar Belaid, M. Jaoua, M. Masmoudi, and M.-L. Siala, *Application of the topological gradient to image restoration and edge detection*, Engineering Analysis with Boundary Elements Journal, vol. 32, pp. 891–899, 2008.

[5] K. Krissian, R. Kikinis, C.-F. Westin, and K. Vosburgh, *Speckle-constrained filtering of ultrasound images*, in 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05), vol. 2, pp. 547–552, 2005.

[6] A. Zang and Y. Fu, *Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces*, Nonlinear Anal., vol. 69, pp. 3629–3636, 2008.