A fixed-point iteration for nonlinear PDE involving variable exponent function

Hamdi Braiek*

ESPRIT School of Engineering, 18 rue de l'Usine Charguia II 2035 Ariana, Tunisia &
University of Tunis El Manar, National Engineering School at Tunis, Laboratory for Mathematical and Numerical Modeling in Engineering Science, B.P. 37, 1002 Tunis-Belvédère, Tunisia.

E-mail: hamdi.houichet@gmail.com

Abstract

We study a nonlinear Dirichlet boundary value problem with the variable exponent $p(x)$-Laplacian and a reaction term. First, we prove the existence and uniqueness of a weak solution by minimizing the associated energy functional. To deal with possible singularities in the operator, we introduce a relaxed energy through a regularization method. We then show that this relaxed functional Γ-converges to the original one, and that the minimizers converge to the true solution. On this basis, we design a fixed point iteration scheme. We prove convergence result of the fixed-point procedure. The theoretical analysis is complemented by a finite element discretization and numerical experiments, which demonstrate the effectiveness, stability, and convergence properties of the proposed scheme.

2020 Mathematics Subject Classification. **35J60**. 35J92, 49J45, 65N30, 65K10.
Keywords. $p(x)$-Laplacian, nonlinear elliptic equations, variable exponent Sobolev spaces, Γ-convergence, regularization techniques, fixed-point iteration, finite element method..

1 Introduction

Nonlinear PDEs with variable exponent $p(x)$-Laplacian play an important role in modern analysis because they can model many real-world systems, such as image processing, electro-rheological fluids, and flow in porous media [1, 2, 6, 16, 17]. A special feature of these equations is that the exponent $p(x)$ changes with position. This variation allows the model to describe spatially dependent behaviors more accurately than equations with a constant exponent. At the same time, the presence of a variable exponent creates non-standard growth conditions, which makes the analysis more difficult. In fact, the $p(x)$-Laplacian operator is more complex than the classical p-Laplacian and cannot be studied with traditional techniques alone. For this reason, specialized mathematical tools are needed to handle such problems.

In this article we are interested in the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u) + \alpha(x)|u|^{p(x)-2}u = f, & \in \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, α belongs to $L^\infty(\Omega)$ and $a_0 := \mathrm{essinf}_\Omega \alpha(x) > 0$ and the source function $f \in L^2(\Omega)$. In this context, $\mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u) =$

*The author has legally changed their name from Hamdi Houichet to Hamdi Braiek. ORCID ID: https://orcid.org/0000-0002-3301-1385

Advanced Studies: Euro-Tbilisi Mathematical Journal 19(2) (2026), pp. 123–140.
DOI: 10.32513/asetmj/193220082619209
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 20 July 2025.
Accepted for publication: 26 November 2025.

$\Delta_{p(x)}u$ is the $p(x)$-Laplacian operator with $p \in C(\overline{\Omega})$ satisfies the condition

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x) \leq 2. \tag{1.2}$$

The study of this PDE is based on variable-exponent Sobolev spaces, written as $W^{1,p(x)}(\Omega)$. These spaces are built for functions whose integrability depends on the position, which makes them suitable for the analysis of problems with variable exponents. They give a strong framework for proving the existence and uniqueness of solutions, thanks to results such as the Poincaré inequality [10, 12]. In this way, they are essential for dealing with the non-standard growth that appears in the $p(x)$-Laplacian. However, nonlinear PDEs with variable exponents remain very challenging. On the one hand, the nonlinearity makes it hard to apply standard techniques. On the other hand, the variable exponent leads to non-uniform growth, which requires special care in the choice of function spaces and estimates. Because of these difficulties, classical methods developed for linear or constant-exponent PDEs are often not enough.
To address these challenges, several techniques are employed. One common approach is the use of fixed-point iterations, which transform the nonlinear PDE into a sequence of simpler linear problems [9, 18, 19, 20]. Another approach is regularization, which helps to smooth out singularities or degeneracies and makes the problem more stable [14, 23]. A further tool is Gamma-convergence provides a variational approach to relax the problem and simplify its study. Gamma-convergence is an key tool in the calculus of variations [5]. It is widely used to study the asymptotic behavior of functionals [3, 13]. In the context of PDEs, Gamma-convergence is useful because it allows the relaxation of the original problem into a simpler one that still captures its main features. This relaxation is essential for proving the existence of solutions and also provides a solid basis for the study of numerical methods.
In this work, we focus on Gamma-convergence and fixed point iterations. We reformulate the weak formulation of the relaxed PDE associated to (1.1) as a fixed point problem and then starting with an initial guess, we solve the linearized problem iteratively until the convergence to the solution. This approach leverages standard numerical tools for linear PDEs. It is both efficient and practical.

The paper is organized as follows. Section 2 introduces variable-exponent Sobolev spaces. It reviews key properties and results. Section 3 formulates the Dirichlet problem and presents the existance and uniqueness of the solution. Section 4 discusses Gamma-convergence and its role in relaxation. Section 5 details the fixed point iteration method. It includes the algorithm and convergence analysis. Section 6 presents the finite element discretization together with two numerical experiments that illustrate the behaviour of the relaxed formulation and confirm the convergence properties of the fixed-point iteration.

2 Background

In this section, we present essential concepts and results from Lebesgue and Sobolev spaces with variable exponent functions. For a more detailed background and properties of these spaces, we refer the reader to [8, 10, 12].

During this work, we will assume that the variable-exponent function $p(\cdot)$ satisfies condition (1.2). Let first define $L^{p(\cdot)}(\Omega)$ as set of all the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \tau > 0 \,|\, \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} \mathrm{d}x \leq 1 \right\}$$

is finite. $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is separable and reflexive Banach space [12].

Lemma 2.1. ([12]) Let $p(x), r(x) \in C_+(\overline{\Omega})$. We have

$$\text{if } 1 < p(x) \leq r(x) \text{ for } x \in \overline{\Omega}, \text{ then } L^{r(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega),$$

and the embedding is continuous.

Similarly, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \mid \nabla u \in L^{p(\cdot)}(\Omega)\}.$$

The space $W^{1,p(\cdot)}(\Omega)$, which is equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

is a separable and reflexive Banach space [8]. Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^1(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable, reflexive, and uniformly convex Banach space (see [12, Theorem 2.1]). For $u \in W_0^{1,p(x)}(\Omega)$, we define an equivalent norm

$$\|u\| = \|\nabla u\|_{1,p(x)}.$$

Since the Poincaré inequality holds [8], i.e., there exists a positive constant $C_P > 0$ such that

$$\|u\|_{p(x)} \leq C_P \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega). \tag{2.1}$$

Moreover, for $p(x) \leq N$, we have the compact embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega),$$

if $q \in C^+(\Omega)$ with $q(x) < p^*(x)$ for all $x \in \Omega$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

According to [12, Theorem 1.3], we have

Proposition 2.2. For any $u \in W^{1,p(\cdot)}(\Omega)$, we define the potential $\rho(u) = \displaystyle\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x$. Then, we have

$$\|\nabla u\|^{p-}_{L^{p(\cdot)}(\Omega)} \leq \rho(u) \leq \|\nabla u\|^{p+}_{L^{p(\cdot)}(\Omega)}, \text{ if } \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \geq 1,$$

$$\|\nabla u\|^{p+}_{L^{p(\cdot)}(\Omega)} \leq \rho(u) \leq \|\nabla u\|^{p-}_{L^{p(\cdot)}(\Omega)}, \text{ if } \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \leq 1.$$

3 Existence and uniqueness of solution

In the sequel, we will establish the existence and uniqueness of a solution for the PDE (1.1) in $W_0^{1,\,p(\cdot)}(\Omega)$.

Let as define the energy functional

$$F(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{\alpha(x)}{p(x)}|u|^{p(x)}\,dx - \int_\Omega fu\,dx. \tag{3.1}$$

Proposition 3.1. For fixed $f \in L^2(\Omega)$, the functional (3.1) admits a unique solution u in $W_0^{1,p(\cdot)}(\Omega)$. Moreover, the solution u is exactly the solution to the nonlinear PDE (1.1).

Proof. We need to show that $F(u) \to \infty$ as $\|u\| \to \infty$. Since $1/p(x) \geq 1/p^+ \geq 1/2$, and for $\|\nabla u\|_{p(\cdot)} > 1$, Proposition 2.2 gives

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx \geq \frac{1}{p^+}\|\nabla u\|_{p(\cdot)}^{p^-}.$$

Since $\alpha(x) \geq a_0 > 0$, $1/p(x) \geq 1/p^+$, and by the Poincaré inequality (2.1), $\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}$. If $\|u\|_{p(\cdot)} > 1$, according to [12, Theorem 1.3] then

$$\int_\Omega |u|^{p(x)}\,dx \geq \|u\|_{p(\cdot)}^{p^-},$$

so

$$\int_\Omega \frac{\alpha(x)}{p(x)}|u|^{p(x)}\,dx \geq \frac{a_0}{p^+}\int_\Omega |u|^{p(x)}\,dx \geq \frac{a_0}{p^+}C'\|\nabla u\|_{p(\cdot)}^{p^-}.$$

Since $f \in L^2(\Omega)$, we can easily estimate

$$\left|\int_\Omega fu\,dx\right| \leq \|f\|_{L^2}\|u\|_{L^2} \leq C\|f\|_{L^2}\|\nabla u\|_{L^{p(\cdot)}}.$$

Using Young's inequality, for any $\varepsilon > 0$, we obtain

$$\left|\int_\Omega fu\,dx\right| \leq \frac{\varepsilon}{p^+}\|\nabla u\|_{p(\cdot)}^{p^+} + C(\varepsilon, p^+)\|f\|_{L^2}^{\frac{p^+}{p^+-1}}.$$

Therefore, we get

$$F(u) \geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{a_0}{p^+}\int_\Omega |u|^{p(x)}\,dx - \frac{\varepsilon}{p^+}\|\nabla u\|_{p(\cdot)}^{p^+} - C\|f\|_{L^2}^{\frac{p^+}{p^+-1}}.$$

By Proposition 2.2, and $p(x) \leq p^+$

$$F(u) \geq \frac{1-\varepsilon}{p^+}\|\nabla u\|_{p(\cdot)}^{p^-} + \frac{a_0}{p^+}C'\|\nabla u\|_{p(\cdot)}^{p^-} - C\|f\|_{L^2}^{\frac{p^+}{p^+-1}}.$$

Choose $\varepsilon < 1$, and since $p^- > 1$, $F(u) \to +\infty$ as $\|\nabla u\|_{p(\cdot)} \to \infty$. Hence, F is coercive.

The functional $F(u)$ is the sum of three terms the first two terms are convex and continuous, hence weakly lower semicontinuous in $W_0^{1,p(\cdot)}(\Omega)$. The last term is linear and continuous, hence weakly lower semicontinuous. Therefore, F is weakly lower semicontinuous in $W_0^{1,p(\cdot)}(\Omega)$. Additionally, since $W_0^{1,p(\cdot)}(\Omega)$ is reflexive, there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ such that

$$F(u) = \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} F(v).$$

The uniqueness of the minimizer follows from the strict convexity of $F(u)$.

Let u be the minimizer. For any $v \in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned}\langle F'(u), v\rangle &= \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} \\ &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \, dx + \int_\Omega \alpha(x)|u|^{p(x)-2} uv \, dx - \int_\Omega fv \, dx = 0.\end{aligned}$$

Therefore, the functional $F(\cdot)$ is Gâteaux differentiable in $W_0^{1,p(\cdot)}(\Omega)$ and its unique minimizer is a solution of the weak formulation

$$\begin{cases}\text{Find } u \in W_0^{1,p(\cdot)}(\Omega) \text{ such that} \\ \displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \, dx + \int_\Omega \alpha(x)|u|^{p(x)-2} uv \, dx = \int_\Omega fv \, dx, \quad \forall v \in W_0^{1,p(\cdot)}(\Omega).\end{cases} \tag{3.2}$$

By integrating by part and using Green formula, we easily get that u is the unique solution to the PDE (1.1).

Q.E.D.

4 Gamma-convergence relaxation

Note that the terms $|\nabla u|^{p(x)-2}$ and $|u|^{p(x)-2}$ in the PDE (1.1) may become singular when $|\nabla u| = 0$ or $|u| = 0$, respectively, which complicate numerical and analytic treatment. To address these singularities and ensure numerical stability, we introduce a regularized functional $F_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$ for a small parameter $\varepsilon > 0$

$$F_\varepsilon(u) = \int_\Omega \Psi_\varepsilon\left(|\nabla u|^2\right) \, dx + \int_\Omega \alpha(x)\Psi_\varepsilon\left(|u|^2\right) \, dx - \int_\Omega fu \, dx. \tag{4.1}$$

where

$$\Psi_\varepsilon(s) = \frac{1}{p(x)}(\varepsilon + s)^{\frac{p(x)}{2}} + \varepsilon s, \text{ for all } s \geq 0.$$

The term $(\varepsilon+s)^{\frac{p(x)}{2}}$ regularizes $s^{\frac{p(x)}{2}}$ by approximating $|\nabla u|$ with $\sqrt{\varepsilon + |\nabla u|^2}$ and $|u|$ with $\sqrt{\varepsilon + |u|^2}$, ensuring that the integrands are well-defined for all $u \in H_0^1(\Omega)$. The additional term εs enhances coercivity in $H_0^1(\Omega)$, facilitating the existence of minimizers and numerical convergence.

We will establish the ellipticity of the regularized functional (4.1).

Proposition 4.1. Let $p \in C(\bar{\Omega})$ satisfies (1.2), $\alpha(x) \geq a_0 > 0$, and $f \in L^2(\Omega)$. Then, the functional F_ε is coercive in $H_0^1(\Omega)$.

Proof. Let $u \in H_0^1(\Omega)$, by Cauchy-Schwarz and Poincaré inequalities, there exists $C_P > 0$ such that $\|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2}$, so

$$\left| \int_\Omega f u \, dx \right| \leq \|f\|_{L^2} \|u\|_{L^2} \leq C_P \|f\|_{L^2} \|u\|_{H_0^1}.$$

Using the inequality $ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$, set $a = \|u\|_{H_0^1}$, and $b = C_P \|f\|_{L^2}$

$$C_P \|f\|_{L^2} \|u\|_{H_0^1} \leq \frac{\varepsilon}{2} \|u\|_{H_0^1}^2 + \frac{1}{2\varepsilon} C_P^2 \|f\|_{L^2}^2.$$

Since $p(x) \leq p^+ \leq 2$, we have $\left(\varepsilon + |\nabla u|^2\right)^{\frac{p(x)}{2}} \geq \varepsilon^{\frac{p(x)}{2}} \geq \varepsilon^{\frac{p^+}{2}} > 0$ and similarly for $\left(\varepsilon + |u|^2\right)^{\frac{p(x)}{2}}$. Also, $\alpha(x) \geq a_0 > 0$, we get

$$\begin{aligned} F_\varepsilon(u) &\geq \varepsilon \|u\|_{H_0^1}^2 + \varepsilon \|u\|_{L^2}^2 + \frac{\varepsilon^{\frac{p^+}{2}}}{p^+} (1 + a_0) |\Omega| - C_P \|f\|_{L^2} \|u\|_{H_0^1} \\ &\geq \frac{\varepsilon}{2} \|u\|_{H_0^1}^2 + \frac{\varepsilon^{\frac{p^+}{2}}}{p^+} (1 + a_0) |\Omega| - \frac{1}{2\varepsilon} C_P^2 \|f\|_{L^2}^2 \to \infty \text{ as } \|u\|_{H_0^1}^2 \to \infty. \end{aligned}$$

Q.E.D.

Lemma 4.2. For each $\varepsilon > 0$, F_ε has at least one minimizer in $H_0^1(\Omega)$, i.e., there exists $u_\varepsilon \in H_0^1(\Omega)$ such that

$$F_\varepsilon(u_\varepsilon) = \inf_{u \in H_0^1(\Omega)} F_\varepsilon(u).$$

We have the following proprities.

Proposition 4.3. Let $\Psi_\varepsilon(s) = \frac{1}{p(x)} (\varepsilon + s)^{\frac{p(x)}{2}} + \varepsilon s$, where $\varepsilon > 0$, $s \in \mathbb{R}_+$, and $p \in C(\overline{\Omega})$ with $1 < p^- \leq p(x) \leq p^+ \leq 2$. Define $\psi_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ by $\psi_\varepsilon(s) := \Psi_\varepsilon'(s) = \frac{1}{2} (\varepsilon + s)^{\frac{p(x)}{2} - 1} + \varepsilon$. Then the following properties hold

(a) There exist constants $\alpha_0, \alpha_1 > 0$ such that

$$\alpha_0 \leq \psi_\varepsilon(s) \leq \alpha_1, \quad \forall s \in \mathbb{R}_+, \forall x \in \overline{\Omega}.$$

(b) The function ψ_ε is nonincreasing

$$\psi_\varepsilon'(s) \leq 0, \quad \forall s \in \mathbb{R}_+, \forall x \in \overline{\Omega}.$$

(c) There exists $\alpha_2 > 0$ such that

$$\psi_\varepsilon(s) + 2s \psi_\varepsilon'(s) \geq \alpha_2, \quad \forall s \in \mathbb{R}_+.$$

(d) There exists $\alpha_3 > 0$ such that, for all $\xi, X \in \mathbb{R}^N$ and $x \in \overline{\Omega}$,

$$\left(\left[\psi_\varepsilon(|\xi|^2)I + 2\psi_\varepsilon'(|\xi|^2)\xi \otimes \xi\right] X, X\right) \geq \alpha_3 |X|^2.$$

Proof. (a) Since $p(x) \in [p^-, 2]$, we have $\frac{p(x)}{2} \in \left[\frac{p^-}{2}, 1\right]$, so $\frac{p(x)}{2} - 1 \in \left[\frac{p^-}{2} - 1, 0\right]$. Thus, the exponent $\frac{p(x)}{2} - 1 \leq 0$, and the function $s \mapsto (\varepsilon + s)^{\frac{p(x)}{2} - 1}$ is nonincreasing on $\mathbb{R}_+$. As a result, $\psi_\varepsilon(s)$ is decreasing with

$$\lim_{s \to 0^+} \psi_\varepsilon(s) = \frac{1}{2}\varepsilon^{\frac{p(x)}{2} - 1} + \varepsilon =: \psi_{\max}, \quad \lim_{s \to \infty} \psi_\varepsilon(s) = \varepsilon =: \psi_{\min}.$$

Thus, for all $s \geq 0$,

$$\varepsilon \leq \psi_\varepsilon(s) \leq \frac{1}{2}\varepsilon^{\frac{p^-}{2} - 1} + \varepsilon,$$

and the constants $\alpha_0 = \varepsilon$, $\alpha_1 = \frac{1}{2}\varepsilon^{\frac{p^-}{2} - 1} + \varepsilon$ satisfy the claim.

(b) Compute the derivative of ψ_ε

$$\psi_\varepsilon'(s) = \frac{1}{2}\left(\frac{p(x)}{2} - 1\right)(\varepsilon + s)^{\frac{p(x)}{2} - 2}.$$

Since $\frac{p(x)}{2} - 1 < 0$, it follows that $\psi_\varepsilon'(s) < 0$ for all $s > 0$. When $p(x) = 2$, $\frac{p(x)}{2} - 1 = 0$, so $\psi_\varepsilon'(s) = 0$, and $\psi_\varepsilon(s) = \frac{1}{2} + \varepsilon$ is constant. Thus, ψ is nonincreasing, satisfying the claim. (c) For all $s \in \mathbb{R}_+$, we have

$$\begin{aligned}
\psi_\varepsilon(s) + 2s\psi_\varepsilon'(s) &= \frac{1}{2}(\varepsilon + s)^{\frac{p(x)}{2} - 1} + \left(\frac{p(x)}{2} - 1\right) s(\varepsilon + s)^{\frac{p(x)}{2} - 2} + \varepsilon \\
&= \frac{1}{2}(\varepsilon + s)^{\frac{p(x)}{2} - 2}\left(\varepsilon + s(p(x) - 1)\right) + \varepsilon.
\end{aligned}$$

Since $p(x) > 1$, all terms are strictly positive for $s \geq 0$. Moreover, the function is continuous and bounded below by a positive constant. Evaluate the infimum

$$\alpha_2 = \inf_{x \in \overline{\Omega}, s \geq 0} \left[\frac{1}{2}(\varepsilon + s)^{\frac{p(x)}{2} - 2}\left(\varepsilon + s(p(x) - 1)\right) + \varepsilon\right].$$

At $s = 0$, $\psi_\varepsilon(0) + 2 \cdot 0 \cdot \psi_\varepsilon'(0) = \frac{1}{2}\varepsilon^{\frac{p(x)}{2} - 1} + \varepsilon$.
As $s \to \infty$, since $\frac{p(x)}{2} - 2 < -1$, the first term vanishes, so $\psi_\varepsilon(s) + 2s\psi_\varepsilon'(s) \to \varepsilon$.
Since $p(x) \in [p^-, 2]$, the minimum occurs at $s \to \infty$, giving $\alpha_2 = \varepsilon > 0$. Thus, the inequality holds.

(d) Define

$$H(x, \xi) := \psi_\varepsilon(|\xi|^2)I + 2\psi_\varepsilon'(|\xi|^2)\xi \otimes \xi.$$

For $X \in \mathbb{R}^N$

$$(H(x, \xi)X, X) = \psi_\varepsilon(|\xi|^2)|X|^2 + 2\psi_\varepsilon'(|\xi|^2)(\xi \cdot X)^2.$$

Since $(\xi \cdot X)^2 \leq |\xi|^2 |X|^2$, we have

$$(H(x,\xi)X, X) \geq \left[\psi_\varepsilon(|\xi|^2) + 2\psi'_\varepsilon(|\xi|^2)|\xi|^2\right] |X|^2.$$

From part (c), $\psi_\varepsilon(|\xi|^2) + 2|\xi|^2\psi'_\varepsilon(|\xi|^2) \geq \alpha_2$. Thus

$$(H(x,\xi)X, X) \geq \alpha_2 |X|^2.$$

Set $\alpha_3 = \alpha_2 = \varepsilon > 0$. The inequality holds for all $\xi, X \in \mathbb{R}^N$, $x \in \overline{\Omega}$, proving the coercivity of the Hessian. Q.E.D.

A minimizer of F_ε satisfies the following Euler-Lagrange equation

$$\begin{cases} -\mathrm{div}\left(2\psi_\varepsilon(|\nabla u|^2)\nabla u\right) + 2\alpha(x)\psi_\varepsilon(|u|^2)u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Remark 4.4. The Euler-Lagrange equation (4.2) associated with the functional F_ε is well-defined in the strong form for $u \in H^2(\Omega) \cap H^1_0(\Omega)$. This follows from the regularity of $\psi_\varepsilon(s)$, which is continuous and bounded for $s \geq 0$, $\varepsilon > 0$, and p satisfies (1.2), as established in property (a) of Proposition 4.3.

The weak form of the equation,

$$\begin{cases} \text{Find } u \in H^1_0(\Omega) \text{ such that} \\ a(u; u, v) = l(v) \quad \forall v \in H^1_0(\Omega), \text{ where,} \\ \quad a(w; u, v) = \displaystyle\int_\Omega 2\psi_\varepsilon(|\nabla w|^2)\nabla u \cdot \nabla v \, dx + \int_\Omega 2\alpha(x)\psi_\varepsilon(|w|^2)uv \, dx, \\ \qquad\quad l(v) = \displaystyle\int_\Omega fv \, dx \quad \text{for a fixed } w \in H^1_0(\Omega). \end{cases} \tag{4.3}$$

is well-defined and elliptic in $H^1_0(\Omega)$, which follow from the boundedness of $\psi_\varepsilon(|\nabla u|^2)$ and $\psi_\varepsilon(|u|^2)$. Thus, $a(u; u, u) \geq 2\varepsilon \|u\|^2_{H^1_0}$, which confirms uniform ellipticity and coercivity in $H^1_0(\Omega)$.

In the sequal, we will show that the functional F_ε approximates F as $\varepsilon \to 0$. Specifically, F_ε Γ-converges to F in the strong topology of $H^1_0(\Omega)$ [5]. This result ensures that minimizers u_ε of F_ε converge (up to a subsequence) to a minimizer u of F, which is the weak solution (3.2) to the original PDE.

Proposition 4.5. Let $p \in C(\overline{\Omega})$ satisfy (1.2) and let $\alpha \in L^\infty(\Omega)$ with $\alpha(x) \geq \alpha_0 > 0$. Let $f \in L^2(\Omega)$. Then, F_ε Γ-converges to F in the strong topology of $H^1_0(\Omega)$ as $\varepsilon \to 0$. Moreover, if $u_\varepsilon \in H^1_0(\Omega)$ is a minimizer of F_ε, and $u \in H^1_0(\Omega)$ is the unique minimizer of F, then $u_\varepsilon \to u$ in $H^1_0(\Omega)$ strongly.

Proof. Let $\{u_\varepsilon\} \subset H^1_0(\Omega)$ be a sequence such that $u_\varepsilon \to u$ in $H^1_0(\Omega)$. This implies $\nabla u_\varepsilon \to \nabla u$ in $L^2(\Omega)$, $u_\varepsilon \to u$ in $L^2(\Omega)$, and, since $p(x) \leq 2 < 2^*$ (where $2^* = \frac{2N}{N-2}$ for $N > 2$), the embedding $H^1_0(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact, so $u_\varepsilon \to u$ in $L^{p(x)}(\Omega)$.

We start by the liminf unequality. Since $u_\varepsilon \to u$ in $L^2(\Omega)$ and $f \in L^2(\Omega)$,

$$\int_\Omega f u_\varepsilon \, dx \to \int_\Omega f u \, dx.$$

Since $p(x) \leq 2$, $(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p(x)}{2}} \leq C(|\nabla u_\varepsilon|^2 + \varepsilon)$, and $\{|\nabla u_\varepsilon|^2\}$ is bounded in $L^1(\Omega)$. By the dominated convergence theorem

$$\int_\Omega \Psi_\varepsilon(|\nabla u_\varepsilon|^2)\,dx \to \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx.$$

Similarly for the reaction term for $\alpha \in L^\infty(\Omega)$, $|u_\varepsilon|^2 \to |u|^2$ in $L^1(\Omega)$, and $p(x)/2 \leq 1$, again by dominated convergence

$$\int_\Omega \alpha(x)\Psi_\varepsilon(|u_\varepsilon|^2)\,dx \to \int_\Omega \alpha(x)\frac{1}{p(x)}|u|^{p(x)}\,dx.$$

Thus

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \alpha(x)\frac{1}{p(x)}|u|^{p(x)}\,dx - \int_\Omega fu\,dx = F(u).$$

For any $u \in H_0^1(\Omega)$, we will show the limsup inequality. Choose the constant sequence $u_\varepsilon = u$. Then

$$F_\varepsilon(u) = \int_\Omega \Psi_\varepsilon(|\nabla u|^2)\,dx + \int_\Omega \alpha(x)\Psi_\varepsilon(|u|^2)\,dx - \int_\Omega fu\,dx.$$

Similarly as $\varepsilon \to 0$

$$\Psi_\varepsilon(s) \to \frac{1}{p(x)}s^{p(x)}, \quad \forall s \geq 0$$

By the dominated convergence theorem

$$\lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u).$$

Thus

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u),$$

satisfying the limsup inequality.

Now, let $u_\varepsilon \in H_0^1(\Omega)$ be a minimizer of F_ε, and let $u \in H_0^1(\Omega)$ be the unique minimizer of F. From Proposition 4.1, F_ε is coercive in $H_0^1(\Omega)$, then $\{u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$. By the Banach-Alaoglu theorem, there exists a subsequence (relabeled u_ε) such that $u_\varepsilon \to u^*$ weakly in $H_0^1(\Omega)$, so $\nabla u_\varepsilon \rightharpoonup \nabla u^*$ in $L^2(\Omega)$, and $u_\varepsilon \to u^*$ in $L^2(\Omega)$. Since F_ε Γ-converges to F, and F is coercive and weakly lower semicontinuous if u_ε minimizes F_ε, then

$$F(u^*) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon\to 0} F_\varepsilon(u) = F(u),$$

where the last equality comes from the limsup inequality with the recovery sequence $u_\varepsilon = u$. Since u is the unique minimizer of F we have $u^* = u$.

Since $u_\varepsilon \rightharpoonup u$, and the energy convergence $\lim F_\varepsilon(u_\varepsilon) = F(u)$. Due to the strict convexity of F and the coercivity of F_ε this implies that $\|\nabla u_\varepsilon\|_{L^2} \to \|\nabla u\|_{L^2}$. Since $H_0^1(\Omega)$ is uniformly convex, it follows that $u_\varepsilon \to u$ strongly in $H_0^1(\Omega)$. Q.E.D.

5 Fixed point iteration

In this section we use the fixed point iteration to linearize the nonlinear problem by solving a sequence of linear problems. The fixed point procedure is summurized in Algorithm 1.

Algorithm 1: Fixed point method for solving (4.3)

1. **Initialization**: Select an initial guess $u_0 \in H_0^1(\Omega)$.
2. **Iterative Step**: For $k = 0, 1, 2, \ldots$, given $u_k \in H_0^1(\Omega)$, compute $u_{k+1} \in H_0^1(\Omega)$ such that
$$a(u_k; u_{k+1}, v) = l(v) \quad \forall v \in H_0^1(\Omega), \tag{5.1}$$
where $a(\cdot;\cdot,\cdot)$ and $l(\cdot)$ are given in (4.3).
3. **Convergence Check**: Stop when $\|u_{k+1} - u_k\|_{H_0^1} < \delta$, where $\delta > 0$ is a small tolerance.

Define the functional
$$J(u) = \int_\Omega \Psi_\varepsilon(|\nabla u|^2)\,dx + \int_\Omega \alpha(x)\Psi_\varepsilon(|u|^2)\,dx.$$
Since $\Psi''_\varepsilon(t) = \psi'_\varepsilon(t) \le 0$, Ψ_ε is concave. Let $u_k, u_{k+1} \in H_0^1(\Omega)$, we obtain
$$\Psi_\varepsilon(|\nabla u_{k+1}|^2) \le \Psi_\varepsilon(|\nabla u_k|^2) + \psi_\varepsilon(|\nabla u_k|^2)(|\nabla u_{k+1}|^2 - |\nabla u_k|^2). \tag{5.2}$$
$$\Psi_\varepsilon(|u_{k+1}|^2) \le \Psi_\varepsilon(|u_k|^2) + \psi_\varepsilon(|u_k|^2)(|u_{k+1}|^2 - |u_k|^2). \tag{5.3}$$
Multiply (5.3) by $\alpha(x) \ge 0$ summing up and integrating
$$J(u_{k+1}) - J(u_k) \le \frac{1}{2}a(u_k; u_{k+1}, u_{k+1}) - \frac{1}{2}a(u_k; u_k, u_k). \tag{5.4}$$

Theorem 5.1. The sequence of solutions $(u_k)_k$ of the linearized problems (5.1) converges to the unique solution $u \in H_0^1(\Omega)$ of the nonlinear problem (4.3) as $k \to \infty$.

Proof. Let u_{k+1} be the solution to the linearized problem (5.1). Thanks to the coercivity of the bilinear form $a(u_k;\cdot,\cdot)$ in $H_0^1(\Omega)$, there exists $c > 0$ such that
$$\begin{aligned} c\|u_{k+1} - u_k\|^2_{H_0^1(\Omega)} &\le a(u_k; u_{k+1} - u_k, u_{k+1} - u_k) = a(u_k; u_k, u_k) - 2a(u_k; u_{k+1}, u_k) + a(u_k; u_{k+1}, u_{k+1}) \\ &= a(u_k; u_k, u_k) - 2a(u_k; u_{k+1}, u_k) + 2a(u_k; u_{k+1}, u_{k+1}) - a(u_k; u_{k+1}, u_{k+1}) \end{aligned}$$
From (5.1), we get
$$c\|u_k - u_{k-1}\|^2_{H_0^1(\Omega)} \le \big(a(u_k; u_k, u_k) + 2l(u_k)\big) - \big(a(u_k; u_{k+1}, u_{k+1}) + 2l(u_{k+1})\big)$$
By the continuity of the linear form $l(\cdot)$ and Poincaré inequality, we get
$$|l(u_k) - l(u_{k+1})| \le c\|u_k - u_{k+1}\|_{L^2(\Omega)} \le C_P\|u_k - u_{k+1}\|_{H_0^1(\Omega)}. \tag{5.5}$$

Using (5.5) and (5.4)

$$0 \leq c\|u_k - u_{k-1}\|^2_{H^1_0(\Omega)} \leq a(u_k; u_k, u_k) - a(u_k; u_{k+1}, u_{k+1}) + 2C\|u_k - u_{k+1}\|_{H^1_0(\Omega)}$$
$$\leq 2(J(u_k) - J(u_{k+1})) + 2C\|u_k - u_{k+1}\|_{H^1_0(\Omega)}$$

Since $J(u_k)$ is decreasing and bounded from below, it converges. Therefore, the difference $J(u_k) - J(u_{k+1}) \to 0$ as $k \to \infty$, which implies that $\|u_{k+1} - u_k\|_{H^1_0(\Omega)} \to 0$ as $k \to \infty$

Define the nonlinear operator $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ by

$$\langle Au, v\rangle = a(u; u, v).$$

From Proposition 4.3, we can show that A is strongly monotone. Let u be the unique weak solution of (4.3), we obtain

$$\begin{aligned}
\alpha\|u_k - u\|^2_{H^1_0(\Omega)} &\leq \langle Au_k - Au, u_k - u\rangle_2 \\
&= a(u_k; u_k, u_k - u) - a(u; u, u_k - u) \\
&= a(u_k; u_k - u_{k+1}, u_k - u) - a(u; u, u_k - u) + a(u_k; u_k, u_k - u) \\
&= a(u_k; u_k - u_k, u_k - u).
\end{aligned}$$

By the continuity of $a(u_k; \cdot, \cdot)$, we have

$$\|u_k - u\|^2_{H^1_0(\Omega)} \leq c\|u_k - u_{k+1}\|_{H^1_0(\Omega)}\|u_k - u\|_{H^1_0(\Omega)}, \quad c > 0,$$

which implies that u_k converges to u in $H^1_0(\Omega)$.

Q.E.D.

6 Numerical implementation

6.1 Discretization

In this section we present the finite element discretization for the implementation of the linearized problems (5.1) produced by Algorithm 1.

Let $\mathcal{T}_h$ be a family of shape-regular, quasi-uniform triangulations of the polygonal domain Ω. Denote by $h = \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$ the mesh size. Let $\mathbb{P}_1(T)$ be the continuous piecewise linear elements on triangles T. Define the Lagrangian finite element space of degree 1 with homogeneous Dirichlet boundary conditions

$$V_h = \{v_h \in C(\overline{\Omega}) \ : \ v_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h, ; v_h = 0 \text{ on } \partial\Omega\} \subset H^1_0(\Omega).$$

Let $\{\varphi_i\}_{i=1}^{n_h}$ be the nodal basis associate with mesh vertices of $\mathcal{T}_h$. We identify each finite element function $v_h \in V_h$ with its vector of degrees of freedom $V = (v_1, \ldots, v_{n_h}) \in \mathbb{R}^{n_h}$ such that $v_h(x) = \sum_{i=1}^{n_h} v_i\varphi_i(x)$.

Given $u_k^h \in V_h$, the discrete linearized problem associated to (5.1) reads

$$\begin{cases} \text{Find } u_{k+1}^h \in V_h \text{ such that} \\ a(u_k^h; u_{k+1}^h, v_h) = l(v_h) \quad \forall v_h \in V_h, \text{ where,} \\ \quad a(w^h; u^h, v^h) = \displaystyle\int_\Omega 2\psi_\varepsilon(|\nabla w^h|^2)\, \nabla u^h \cdot \nabla v^h \, dx + \int_\Omega 2\alpha(x)\psi_\varepsilon(|w^h|^2)\, u^h v^h \, dx, \\ \qquad\qquad l(v^h) = \displaystyle\int_\Omega f v^h \, dx, \quad \text{for a fixed } w^h \in V_h. \end{cases} \tag{6.1}$$

As in standard variable-coefficient elliptic problems [4, 11], for each fixed w^h the bilinear form is bounded and coercive on $H_0^1(\Omega)$. Thus the discrete problem has a unique solution.

Let u be the exact solution of the regularized continuous problem (4.3) and u^h the Galerkin solution (6.1). By Céa's lemma [7], there exists a constant C independent of h such that the quasi-optimal bound holds

$$\|u - u^h\|_{H_0^1(\Omega)} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H_0^1(\Omega)}.$$

We notice that, for $u \in H^2(\Omega)$, the interpolation estimate gives the optimal convergence rate

$$\|u - u^h\|_{H_0^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}.$$

We now introduce the coefficient functions at the current iterate u_k^h

$$c_k(x) := 2\psi_\varepsilon(|\nabla u_k^h(x)|^2), \quad d_k(x) := 2\alpha(x)\psi_\varepsilon(|u_k^h(x)|^2).$$

The global stiffness matrix $A_k \in \mathbb{R}^{n_h \times n_h}$ and load vector $F \in \mathbb{R}^{n_h}$ have entries

$$(A_k)_{ij} = \int_\Omega c_k(x)\, \nabla\varphi_j \cdot \nabla\varphi_i \, dx + \int_\Omega d_k(x)\, \varphi_j, \varphi_i \, dx, \qquad F_i = \int_\Omega f, \varphi_i \, dx.$$

Thus the algebraic step at iteration k is the symmetric positive definite linear system

$$A_k U_{k+1} = F, \tag{6.2}$$

where $U_{k+1} \in \mathbb{R}^{n_h}$ is the coefficient vector of the FE function u_{k+1}^h. In the implementation, the mapping We therefore introduce fixed method procedures for the discrete problem in Algorithm 2.

Algorithm 2: Discrete fixed-point iteration

---

1. **Initialization.** Choose $\varepsilon > 0$. Choose an initial vector $U_0 \in \mathbb{R}^{n_h}$ and set $u_0^h(x) = \sum_i (U_0)_i \varphi_i(x)$.
2. **Iteration.** For $k = 0, 1, 2, \ldots$ until convergence
   - Compute $c_k(x)$ and $d_k(x)$ from the function u_k^h.
   - Assemble the matrix A_k and the vector F.
   - Solve the system (6.2)
   - Set $u_{k+1}^h = \sum_i (U_{k+1})_i \varphi_i$.
3. **Stopping criterion.** Stop when $\|u_{k+1}^h - u_k^h\|_{H^1(\Omega)} < \delta$, where $\delta > 0$ is a small tolerance.

We notice that the convergence of the continuous iteration is established in Theorem 5.1. The discrete iteration inherits the same contractive structure [21]. As a consequence, the discrete sequence $(u_k^h)_{k\geq 0}$ converges to the unique Galerkin solution u^h, with at least linear convergence [15, 22].

6.2 Numerical examples

In this subsection we illustrate the behaviour of the proposed finite element discretization and the fixed-point linearization through two representative examples. The first experiment involves a discontinuous exponent in order to assess the robustness of the method across an interface where the diffusion properties change abruptly. The second example considers a smooth variable exponent, which allows us to examine the performance of the algorithm in the absence of discontinuities and to highlight the influence of a spatially smooth nonlinearity. In both cases the stopping criterion for the fixed-point iteration produced by Algorithm 2 is terminated once the relative difference between two successive iterates satisfies

$$\mathrm{RelErr}_k = \frac{\|u_{k+1}^h - u_k^h\|_{H^1(\Omega)}}{\|u_{k+1}^h\|_{H^1(\Omega)}} \leq 10^{-5}.$$

The regularization parameter is fixed to $\varepsilon = 10^{-4}$.

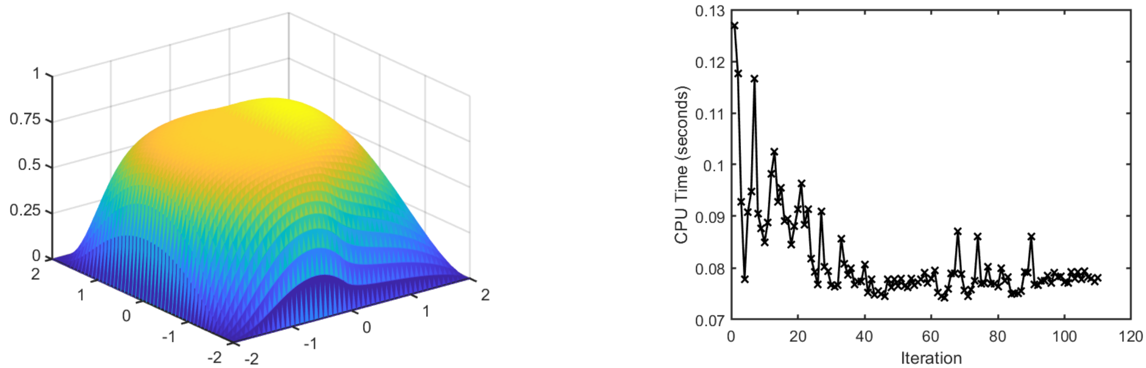
6.2.1 Example 1: Piecewise constant exponent

In this first experiment we consider the square domain $\Omega = [-2, 2]^2$ and set $\alpha \equiv 0$. The exponent $p(x)$ is taken to be piecewise constant, with

$$p(x) = \begin{cases} 1.1 & \text{if } x \leq 0, \\ 1.9 & \text{if } x > 0. \end{cases}$$

p introduces a jump of the diffusion characteristics across the vertical axis. The discrete problem is solved on a sequence of uniform meshes of sizes $h = 0.08$, $h = 0.04$ and $h = 0.02$.

Figure 1 displays the numerical solution obtained on the mesh with $h = 0.08$ together with the corresponding evolution of the cumulative CPU time as the iterations proceed. The computed solution appears smooth on each side of the interface $x = 0$, with a noticeable change of slope across the discontinuity of the exponent. The convergence behaviour of the iteration is further illustrated in Figure 2, where we report the decay of the relative error in logarithmic scale for the three mesh sizes under consideration. All curves exhibit a monotone decrease until the stopping threshold is reached, and their slopes remain comparable across different meshes. This observation indicates that the contraction properties of the fixed-point mapping are essentially independent of the discretization parameter h for this example. The almost linear decay in the semi-logarithmic representation is in agreement with the theoretical expectation that the method converges at least linearly, and the uniformity of the curves across the different meshes confirms the robustness of the approach in the presence of a discontinuous exponent. The computational cost associated with each mesh is summarized in Table 1. The number of iterations required to satisfy the stopping condition remains nearly constant across all refinements, ranging from 110 to 119 iterations. In contrast, the elapsed CPU time increases by approximately a factor of four when the mesh size is



(a) Approximate solution u^h (mesh $h = 0.08$).

(b) Accumulated CPU time vs. iteration (mesh $h = 0.08$).

Figure 1. Example 1: computed solution and CPU-time curve produced by Algorithm 2.

halved, which is consistent with the growth in the number of degrees of freedom and the associated increase in the cost of assembling and solving the discrete linear systems. The combination of a nearly mesh-independent iteration count and an expected scaling of the computational time reflects the good stability and efficiency of the numerical scheme.

Table 1. Convergence results for different mesh sizes on $\Omega = [-2, 2] \times [-2, 2]$

Mesh size h	$n_h \times n_h$	**Element vertices**	**Iterations**	**Elapsed time (s)**
0.08	50×50	10 000	110	9.04
0.04	100×100	40 000	117	37.14
0.02	200×200	160 000	119	151.19

6.2.2 Example 2: Smooth varying exponent

In this second experiment we consider the rectangular domain $\Omega = [0, 1] \times [-1, 1]$. The exponent is chosen to vary smoothly according to

$$p(x) = 1 + \exp\bigl(-(x_1^2 + x_2^2)\bigr)\,,$$

so that $p(x)$ ranges from values close to 1 away from the origin to a maximum value of 2 at the centre of the domain. The coefficient $\alpha(x) = 1$. The discretization is performed on a uniform mesh of size $h = 0.04$, and Algorithm 2 is applied starting from a zero initial guess. The stopping condition is the same as in Example 1, namely the termination of the fixed-point procedure once the relative difference between successive iterates falls below the prescribed tolerance. For this configuration, the method converges in 21 iterations, and the total computational time is approximately 6.74 seconds. The numerical results are presented in Figure 3. The Figure 3(a) displays the computed finite-element solution. The smooth spatial variation of the exponent produces a correspondingly smooth solution profile, in contrast with the slope discontinuity observed in Example 1 where

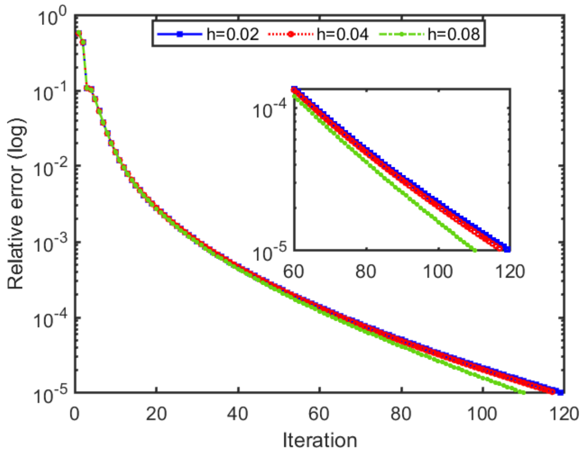
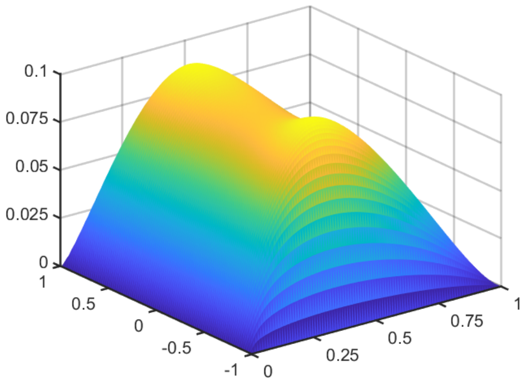


FIGURE 2. Example 1: relative error RelErr_k ($\log_{10}$ scale) versus iteration for mesh sizes $h = 0.02, 0.04, 0.08$.

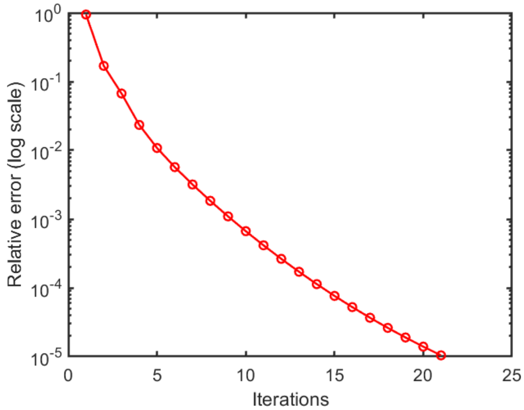
$p(x)$ exhibited a jump. No sharp layer or loss of regularity is noticeable, confirming that the fixed-point linearization handles smoothly varying exponents without difficulty. The Figure 3(b) reports the evolution of the relative error RelErr_k in base-10 logarithmic scale. The error decreases monotonically from the first iteration onward and exhibits a nearly linear decay, illustrating again the contractive nature of the iteration. The Figure 3(c) displays the cumulative CPU time as a function of the iteration count. The curve decreases steadily and almost proportionally to the number of iterations, reflecting the uniform cost of assembling and solving the discrete linear system at each step.

Conclusion

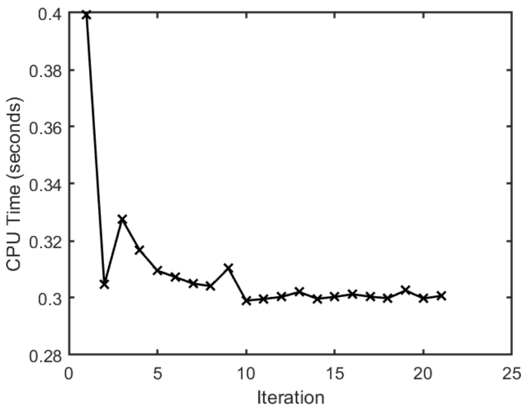
We studied a nonlinear elliptic problem involving the variable-exponent $p(x)$-Laplacian with a reaction term. We established the existence and uniqueness of a weak solution by minimizing the associated energy functional. To handle possible singularities, we introduced a regularized formulation and proved that it Γ-converges to the original functional, thereby ensuring consistent approximations. We then developed a fixed-point linearization scheme and demonstrated its strong convergence to the unique solution. In the numerical implementation, finite element discretization combined with the proposed iteration was employed. Two numerical experiments validated the theoretical results and confirmed the robustness, stability, and efficiency of the method, even in the presence of discontinuous or strongly varying exponents.



(a) Approximate solution u^h (mesh $h = 0.04$).



(b) Relative error in logarithmic scale versus iteration.



(c) Accumulated CPU time as a function of the iteration count.

FIGURE 3. Example 2: computed solution, CPU-time curve produced by Algorithm 2 and relative error RelErr_k ($\log_{10}$ scale) versus iteration for mesh size $h = 0.04$.

References

[1] E. Acerbi, G. Mingione, and G. Seregin. *Regularity results for parabolic systems related to a class of non-Newtonian fluids.* In Annales de l'Institut Henri Poincaré C, Analyse non linéaire, volume 21, pages 25–60. Elsevier, 2004.

[2] S. N. Antontsev and J. F. Rodrigues. *On stationary thermo-rheological viscous flows.* Annali-Universita Di Ferrara Sezione 7, 52(1):19, 2006.

[3] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces: applications to PDEs and optimization.* SIAM, 2014.

[4] D. Braess. *Finite elements: Theory, Fast Solvers, and Applications in Solid Mechanics.* Cambridge University Press, 2001.

[5] A. Braides. *Gamma-convergence for Beginners*, volume 22. Clarendon Press, 2002.

[6] H. Braiek. *Primal-dual method for image denoising with variable exponent Sobolev spaces.* Moroccan Journal of Pure and Applied Analysis, 11(2):117–133, 2025.

[7] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* SIAM, 2002.

[8] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka. *Lebesgue and Sobolev Spaces with Variable Exponents.* Springer, 2011.

[9] G. Dinca. *A fixed point method for the $p(\cdot)$-Laplacian.* Comptes Rendus. Mathématique, 347(13-14):757–762, 2009.

[10] D. E. Edmunds and J. Rákosník. *Sobolev embeddings with variable exponent.* Studia Mathematica, 143(3):267–293, 2000.

[11] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159. Springer, 2004.

[12] X. Fan and D. Zhao. *On the spaces $L^{p(x)}$ and $W^{m,p(x)}$.* Journal of Mathematical Analysis and Applications, 263(2):424–446, 2001.

[13] M. d. M. González. *Gamma convergence of an energy functional related to the fractional Laplacian.* Calculus of variations and partial differential equations, 36(2):173–210, 2009.

[14] K. Ho and I. Sim. *Existence results for degenerate $p(x)$-Laplace equations with Leray-Lions type operators.* Science China Mathematics, 60(1):133–146, 2017.

[15] M. Holst, G. Tsogtgerel, and Y. Zhu. *Local convergence of adaptive methods for nonlinear partial differential equations.* arXiv preprint arXiv:1001.1382, 2010.

[16] H. Houichet, A. Theljani, and M. Moakher. *A nonlinear fourth-order PDE for image denoising in Sobolev spaces with variable exponents and its numerical algorithm.* Computational and Applied Mathematics, 40(3):70, 2021.

[17] H. Houichet, A. Theljani, B. Rjaibi, and M. Moakher. *A nonstandard higher-order variational model for speckle noise removal and thin-structure detection.* Journal of Mathematical Study, 52(4):394–424, 2021.

[18] M. Kallel, M. Moakher, and A. Theljani. *The Cauchy problem for a nonlinear elliptic equation: Nash-game approach and application to image inpainting.* Inverse Problems and Imaging, 9(3):853–874, 2015.

[19] J. Moser. *A rapidly convergent iteration method and non-linear partial differential equations-I.* Annali della Scuola Normale Superiore di Pisa - Scienze Fisiche e Matematiche, 20(2):265–315, 1966.

[20] S. Pereverzyev, R. Pinnau, and N. Siedow. *Regularized fixed-point iterations for nonlinear inverse problems.* Inverse Problems, 22(1):1, 2005.

[21] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, volume 49 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[22] M. Tatari, M. Kamranian, and M. Dehghan. *The finite point method for the p-Laplace equation.* Computational Mechanics, 48(6):689–697, 2011.

[23] A. Theljani. *Multi-scale non-standard fourth-order PDE in image denoising and its fixed point algorithm.* International Journal of Numerical Analysis & Modeling, 18(1), 2021.