Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/xx.x.xx

Optimal Control Problem with Total Variation for $p(x)$-Biharmonic Constraint and Approximation

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Received: May 18, 2025 Accepted:

Abstract: We study an optimal control problem combining total variation–regularized controls with a variable-exponent biharmonic operator. Such models arise in the optimal design of heterogeneous elastic structures and in adaptive regularization problems in image processing. For each admissible control, existence and uniqueness of weak solutions to the associated state equation are established, together with the existence of an optimal control. To overcome the degeneracy of the nonlinear $p(x)$-biharmonic operator, an ε-relaxed formulation is introduced, restoring strict ellipticity. The convergence of optimal solutions of the relaxed problems toward those of the original formulation is analyzed using Γ-convergence techniques. Numerical experiments based on finite element discretization, combined with fixed-point iteration and a primal–dual algorithm, illustrate the applicability of the proposed approach and confirm the theoretical convergence results.

Keywords: Optimal control; Total variation; variable exponent; $p(x)$-biharmonic PDE; Nonlinear elliptic equation; Optimization problem.

2010 Mathematics Subject Classification. 35J70; 35J60; 49J20; 93C73; 49J45.

1 Introduction

Control theory involving constraints given by partial differential equations (PDEs) is an important field of study. It has many applications in engineering, physics, and applied mathematics. Among these equations, higher-order PDEs like the biharmonic equation are especially useful. They are used to describe real-world problems. For example, fourth-order model the flexural behavior of thin elastic plates under bending loads in continuum mechanics and elasticity theory [5, 26]. When material stiffness or bending resistance is not uniform across the domain, as in functionally graded materials or engineered composites, models with spatially dependent exponents or coefficients better capture the heterogeneous response of the structure. In such settings, optimizing the spatial distribution of material stiffness against performance criteria e.g., minimal deflection under load or maximal stiffness for a given mass leads naturally to control-of-coefficient problems for high-order operators. In image processing, variable-exponent and mixed growth functionals have been proposed to provide spatially adaptive regularization to reduce smoothing near edges [13, 14, 24, 25], and TV-type weights are commonly used to preserve sharp transitions [10, 11, **?**]; such modeling motivations justify the combination of TV control and a spatially varying exponent in our cost and state formulations. Similarly, models for variable-density or spatially heterogeneous porous flows use PDEs with space-dependent coefficients and nonstandard growth to capture changing rheology or permeability [**?**, 19]. Our focus in this work is on an optimal control problem (OCP) governed by a quasilinear $p(x)$-biharmonic PDE on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ (typically $N = 2$ or 3). Specifically, we aim to

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minimize the functional

$$\min_{(\upsilon,y)} \mathscr{J}(\upsilon,y), \quad \text{with } \mathscr{J}(\upsilon,y) = \int_\Omega |y-y_d|^{p(x)} dx + TV(\upsilon), \tag{1}$$

subject to the $p(x)$-biharmonic equation:

$$\begin{cases} \Delta(\upsilon|\Delta y|^{p(x)-2}\Delta y) = f, & \text{in} \quad \Omega \\ y = \frac{\partial y}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where the exponent $p(x)$ is a log-Hölder function that satisfies

$$p(x) \in C(\overline{\Omega}), \quad 2 \le p^- \le p(x) \le p^+ < \infty. \tag{3}$$

$y_d \in L^{p(x)}$ is a desird target profile, $f \in L^{p'(x)}(\Omega)$ is a given source term, and $p'(x)$ is the Hölder conjugate function of p. The functionnal $TV(\upsilon)$, in the minimization problem (1), is the total variation regularization of a control υ which penalize oscillations, it defined by

$$TV(\upsilon) = \sup_{\mathfrak{g}\in C_c^1(\Omega;\mathbb{R}^N), \|\mathfrak{g}(x)\|_\infty \le 1} \int_\Omega \upsilon(x)\operatorname{div}\mathfrak{g}(x)\,dx,$$

and $BV(\Omega) = \left\{\upsilon \in L^1(\Omega) \middle| D\upsilon \in \mathscr{M}(\Omega;\mathbb{R}^N), TV(\upsilon) < +\infty\right\}$ is the space of bounded variation.

The study of PDEs with variable exponents has garnered significant attention in recent years due to their ability to model complex phenomena in real sciences. In [14,21,23], the authors explored the Lebesgue and Sobolev spaces with variable exponent and their applications to PDEs. In [1,2], the authors addressed $p(x)$-biharmonic problems, providing insights into existence and non-existence of solutions. In optimal control (OC), total variation regularization for controls in bounded variation spaces has been studied in various contexts [28], but its application to variable-exponent biharmonic equations remains a novel contribution of this work. In fact, very few works address even the simpler variable-exponent Laplace equation in an OC context. For instance, D'Apice et al. [17] showed that distributed or boundary control problems for $p(x)$-Laplacian equations are largely open and may be ill-posed, often requiring relaxation techniques. In the constant-exponent case, Casas et al. [9] studied an elliptic p-Laplace equation with a weighted term $\upsilon(x)$, they introduce two-parameter regularization to handle degeneracy, and established that the weak solution exist and unique, as well as the convergence of regularized approximations to the original problem. Also, Kogut et al. [27] analyzed an OCP for a p-biharmonic PDE arising in plate design, using Henig cone relaxation to address pointwise state constraints. However, to our knowledge no previous work has combined a quasilinear biharmonic equation with a variable exponent $p(x)$, where the control term uses bounded variation regularization and includes a relaxation method to guarantee convergence.

The proposed control framework allows one to modulate the local behavior of a fourth-order nonlinear diffusion process through a spatially distributed coefficient. The control variable υ governs the intensity of the bending or diffusion mechanism, while the variable exponent $p(x)$ adapts the nonlinear response of the system to spatial heterogeneities. This interplay provides a flexible modeling tool for problems where the underlying physical or data-driven process exhibits nonuniform structural properties.

The main contributions of this study are summarized as follows:

– We establish, for each admissible control υ, the existence and uniqueness of a weak solution to the quasilinear $p(x)$-biharmonic state equation under mixed Dirichlet–Neumann boundary conditions, together with appropriate regularity results.
– We prove that the control-to-state mapping is well defined on the admissible set $\mathfrak{A}_{ad}$ and show the existence of at least one optimal control υ^* minimizing the proposed cost functional.
– To address the degeneracy of the nonlinear operator $\Delta\left(\upsilon|\Delta y|^{p(x)-2}\Delta y\right)$, we introduce an ε-relaxed state equation that restores strict ellipticity and allows a rigorous analytical treatment.
– We develop the ε-relaxation framework in the Hilbert space $H_0^2(\Omega)$ and analyze the relaxed optimal control problem using Γ-convergence techniques.
– We prove that, as the regularization parameter $\varepsilon \to 0$, the solutions of the relaxed state equation and the corresponding optimal controls converge to those of the original, non-relaxed problem.
– We complement the theoretical analysis with numerical simulations using finite element discretization and iterative optimization schemes, to illustrate the applicability of the proposed framework and the convergence of optimal controls.

The rest of this paper is structured as follows. Section 2 recalls the functional framework and mathematical tools required in the analysis, including variable-exponent Sobolev spaces and functions of bounded variation. Section 3 is devoted to the well-posedness analysis of the state equation and to the proof of existence of OCs within the admissible set $\mathfrak{A}_{ad}$. In Section 4, we introduce an ε-relaxed formulation of the OCP in order to overcome the degeneracy of the nonlinear operator, and we analyze the convergence of relaxed solutions toward solutions of the original problem using Γ-convergence techniques. Section 5 presents the numerical approximation of the relaxed problem, including finite element discretization, the adopted iterative algorithms, and representative numerical experiments illustrating the theoretical results. Finally, We end the article with a conclusion that summarizes the main results and outlines perspectives for future research.

2 Function spaces

We will work in the framework of the Sobolev spaces $W^{k,p(x)}(\Omega)$ and bounded variation spaces $BV(\Omega)$. Key properties of these spaces, will be reviewed.

2.1 Variable exponent Lebesgue and Sobolev spaces

We recall the the fundamental definitions and basic properties of the Lebesgue and Sobolev spaces with variable exponent. For further details, see [15,18]. In the sequel, we assume the exponent $p(x)$ to be log-Hölder continuous and it satisfies (3). In particular, there exists a constant $C>0$ such that

$$|p(x_1)-p(x_2)| \leq \frac{C}{-\log(\|x_1-x_2\|)}, \quad \text{for all } x_1,x_2 \in \Omega \text{ with } \|x_1-x_2\| < \tfrac{1}{2}. \tag{4}$$

The log-Hölder continuity assumption on the variable exponent $p(x)$ plays an important role in the analytical framework. In particular, it guarantees the continuity of the modular and norm in the variable exponent spaces and ensures the density of smooth functions in the variable exponent Lebesgue $L^{p(x)}(\Omega)$ and the variable exponent Sobolev $W^{k,p(x)}(\Omega)$. Moreover, this regularity is essential for the validity of compact embedding results and for the stability of weak convergence under nonlinear operators with $p(x)$-growth [18].

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ consists of all measurable functions $y:\Omega\to\mathbb{R}$ such that the modular $\rho_{p(x)}(y)=\int_\Omega |y(x)|^{p(x)}\,dx<\infty$. This space is equipped with the norm,

$$\|y\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda>0 : \int_\Omega \left|\frac{y(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The space $(L^{p(x)}, \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable and reflexive Banach space, see [29]. We also recall the continuous embedding result.

Theorem 1. *([29]) If we assume $p_1,\ p_2 \in C(\overline{\Omega})$, and $1 \leq p_i^- \leq p_i^+ < \infty$ $(i=1,2)$, are such that $p_1(x) \leq p_2(x)$ for all $x \in \Omega$, then the embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous.*

In addition, for any functions $y \in L^{p(x)}(\Omega)$ and $z \in L^{p'(x)}(\Omega)$, where $1/p(x)+1/p'(x)=1$, the inequality below is valid [29]:

$$\left|\int_\Omega y(x)z(x)\,dx\right| \leq \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}. \tag{5}$$

This ensures that both y and z belong to $L^1(\Omega)$.

Now, we define the variable exponent Sobolev space $W^{k,p(\cdot)}(\Omega)$ as follow:

$$W^{k,p(x)}(\Omega) = \{y \in L^{p(x)}(\Omega) : D^\alpha y \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $D^\alpha y = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}} y$ is the partial derivative of order $|\alpha| = \sum_{i=1}^N \alpha_i$, with $\alpha=(\alpha_1,\ldots,\alpha_N)$ is a multi-index. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)},$$

forms a separable and reflexive Banach space [29].
Furthermore, we can connect these function spaces as follows.

Theorem 2([22]). *Assume that $q \in C(\overline{\Omega};\mathbb{R})$ such that $1 < q^- \leq q^+ < \infty$ and $q(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$ and $k \geq 1$, where*

$$p_k^*(x) = \begin{cases} \frac{N\,p(x)}{N-k\,p(x)} & \text{if } k\,p(x) < N, \\ +\infty & \text{if } k\,p(x) \geq N. \end{cases}$$

Then, the embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous. Moreover, if $q(x) < p_k^*(x)$ the embedding is compact.

We consider $W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$. We then define the space:

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega).$$

According to Theorem 2, the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact, for all $q(x) \leq p_2^*(x) < \infty$. For any function $y \in X$, we define its norm as:

$$\|y\| = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{y(x)}{\lambda}\right|^{p(x)} + \left|\frac{\nabla y(x)}{\lambda}\right|^{p(x)} + \left|\frac{\Delta y(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

According to [**?**], in the space X, the norms $\|\cdot\|_{W^{2,p(x)}}$ and $\|\Delta\cdot\|_{p(x)}$ are equivalent.

For the remainder of this work, we equip the space X with the norm $\|y\|_X := \|\Delta y\|_{L^{p(x)}(\Omega)}$. $(X,\|\cdot\|_X)$ forms a closed subset of $W^{2,p(x)}(\Omega)$, and therefore it is a separable and reflexive Banach space [21].

Similarly as [22], we have the following result.

Proposition 1. *Define the functional $\rho(y) = \int_\Omega |\Delta y|^{p(x)} dx$. For $y \in X$, the following inequalities hold:*

(1) $\|y\|_X < 1$ (respectively $= 1, > 1$) $\Leftrightarrow \rho(y) < 1$ (respectively $= 1, > 1$).
(2) If $\|y\|_X \leq 1$, then $\|y\|_X^{p^+} \leq \rho(y) \leq \|y\|_X^{p^-}$.
(3) If $\|y\|_X \geq 1$, then $\|y\|_X^{p^-} \leq \rho(y) \leq \|y\|_X^{p^+}$.

2.2 Bounded variation space

The space of functions of bounded variation, denoted by $BV(\Omega)$ represents all functions υ in $L^1(\Omega)$ whose distributional derivative $D\upsilon$ is a vector-valued Radon measure with finite total variation i.e., $\upsilon \in \mathscr{M}(\Omega;\mathbb{R}^N)$. More precisely, $\upsilon \in L^1(\Omega)$ belongs to $BV(\Omega)$) if and only if $\upsilon \in L^1(\Omega)$ and there exists a Radon measure $D\upsilon$ satisfying:

$$\int_\Omega \upsilon \operatorname{div}(\mathfrak{g})dx = -\int_\Omega \mathfrak{g}\cdot dD\upsilon, \quad \forall \mathfrak{g} \in C_c^\infty(\Omega;\mathbb{R}^N).$$

By equipping $BV(\Omega)$ with the norm

$$\|\upsilon\|_{BV(\Omega)} = \|\upsilon\|_{L^1(\Omega)} + TV(\upsilon),$$

becomes a Banach space (see [3,20]).

Note that the total variation is lower semicontinuous with respect to convergence in $L^1(\Omega)$ (see [3]). More concretely, we have the result as follow.

Theorem 3([3]). *If $\upsilon_k \in BV(\Omega)$ and such that $\upsilon_k \to \upsilon$ in $L^1(\Omega)$, then*

$$TV(\upsilon) \leq \liminf_{k\to\infty} TV(\upsilon_k).$$

We can now define the weak* convergence by the topology of the BV space.

Theorem 4. *Let υ_k and υ in $BV(\Omega)$. Then, $\upsilon_k \xrightarrow{*} \upsilon^*$ weakly* in $BV(\Omega)$, if and only if $\{\upsilon_k\}$ is bounded in $BV(\Omega)$ and $\upsilon_k \to \upsilon$ in $L^1(\Omega)$.*

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3 Well-posedness of the optimal control problem (1)-(2)

3.1 The elliptic equation (2)

We will show that the PDE involving the $p(x)$-biharmonic operator with Neumann-Dirichlet conditions admits unique solution. First, we define the admissible set $\mathfrak{A}_{ad}$ as:

$$\mathfrak{A}_{ad} = \{\upsilon \in BV(\Omega) \cap L^{\infty}(\Omega) \mid 0 < \alpha \leq \upsilon(x) \leq \beta \text{ a.e. in } \Omega\},$$

Remark. The pointwise bounds $\alpha \leq \upsilon(x) \leq \beta$ are imposed to guarantee uniform ellipticity of the state equation and well-posedness of the associated control-to-state mapping, and they are standard in coefficient control problems where υ represents physically meaningful quantities such as material stiffness, permeability, or regularization strength.

Theorem 5. *Let $p(x)$ satisfy (3) and assume that υ in $BV(\Omega) \cap L^{\infty}(\Omega)$ satisfies $0 < \alpha \leq \upsilon(x) \leq \beta$. If $f \in L^{p'(x)}(\Omega)$, then the equation (2) admits a unique weak solution $y \in X$ satisfying:*

$$\int_{\Omega} \upsilon(x)|\Delta y|^{p(x)-2}\Delta y \Delta\varphi dx = \int_{\Omega} f\varphi dx, \quad \forall \varphi \in X.$$

Moreover, the following estimate holds:

$$\|y\|_X \leq C\|f\|_{L^{p'(x)}}^{1/(p^- -1)}. \tag{6}$$

Proof. Set A as an operator $A : X \to X^*$ defined by:

$$\langle Ay, \varphi\rangle = \int_{\Omega} \upsilon(x)|\Delta y|^{p(x)-2}\Delta y \Delta\varphi\, dx, \quad \forall \varphi \in X.$$

Let $y, z \in X$, we have

$$\langle Ay - Az, y - z\rangle = \int_{\Omega} \upsilon(x) \left(|\Delta y|^{p(x)-2}\Delta y - |\Delta z|^{p(x)-2}\Delta z\right)(\Delta y - \Delta z)\, dx.$$

For $p(x) \geq 2$, the function $t \mapsto |t|^{p(x)-2}t$ is strictly increasing, we get

$$(|t|^{p(x)-2}t - |s|^{p(x)-2}s)(t - s) \geq 0,$$

with equality when $t = s$. Since $\upsilon \geq \alpha > 0$, thus, $\langle Ay - Az, y - z\rangle \geq 0$. Therefore, A is a monotone operator.

If $\langle Ay - Az, y - z\rangle = 0$, since $\upsilon > 0$ then $\Delta y = \Delta z$ a.e. Since $y, z \in X$, with $y = z = 0$ on $\partial\Omega$, and $\frac{\partial y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0$ on $\partial\Omega$, we solve $\Delta(y - z) = 0$ with homogeneous boundary conditions, implying $y = z$. Thus, A is strictly monotone.

Note that, for all $\upsilon(x) \geq \alpha$, we have

$$\langle Ay, y\rangle = \int_{\Omega} \upsilon(x)|\Delta y|^{p(x)}dx \geq \alpha \int_{\Omega} |\Delta y|^{p(x)}dx. \tag{7}$$

By Proposition 1(3), for large $\|\Delta y\|_{L^{p(x)}}$, we can get:

$$\int_{\Omega} |\Delta y|^{p(x)}dx \geq c\|\Delta y\|_{L^{p(x)}}^{p^-}.$$

Thus,

$$\frac{\langle Ay, y\rangle}{\|y\|_X} = \frac{\int_{\Omega} \upsilon|\Delta y|^{p(x)}\, dx}{\|\Delta y\|_{L^{p(x)}}} \geq \alpha c\|\Delta y\|_{L^{p(x)}}^{p^- -1}.$$

since $p^- \geq 2$. Hence, A is coercive.

For $y, z, w \in X$, consider $t \mapsto \langle A(y + tz), w\rangle$:

$$\langle A(y + tz), w\rangle = \int_{\Omega} \upsilon(x)|\Delta(y + tz)|^{p(x)-2}\Delta(y + tz)\Delta w dx.$$

The integrand is continuous in t, and since υ and Δw are bounded appropriately, the dominated convergence theorem ensures continuity in t. Thus, A is hemicontinuous. Now, apply the Minty-Browder theorem for the strictly monotone, coercive, and hemicontinuous operator A in the reflexive space X, there exist a unique $y \in X$ such that $Ay = f$, equivalently to:

$$\langle Ay, \varphi \rangle = \langle f, \varphi \rangle, \quad \text{with } \langle f, \varphi \rangle = \int_\Omega f\varphi dx, \quad \forall \varphi \in X.$$

Now, take $\varphi = y$ in the above formualation, we get:

$$\int_\Omega \upsilon |\Delta y|^{p(x)}\, dx = \int_\Omega f y\, dx.$$

Using Hölder's inequality

$$\left| \int_\Omega f y\, dx \right| \leq \|f\|_{L^{p'(x)}} \|y\|_{L^{p(x)}} \leq \|f\|_{L^{p'(x)}} \|y\|_{W^{2,p(x)}} \leq C \|f\|_{L^{p'(x)}} \|\Delta y\|_{L^{p(x)}}.$$

Together with (7), we get

$$\alpha \int_\Omega |\Delta y|^{p(x)} dx \leq C \|f\|_{L^{p'(x)}} \|\Delta y\|_{L^{p(x)}}.$$

Using $\|\Delta y\|^{p^-}_{L^{p(x)}} \leq \frac{1}{c} \int_\Omega |\Delta y|^{p(x)} dx$, we obtain:

$$\alpha \|\Delta y\|^{p^-}_{L^{p(x)}} \leq C' \|f\|_{L^{p'(x)}} \|\Delta y\|_{L^{p(x)}}.$$

If $\|\Delta y\|_{L^{p(x)}} \neq 0$, we get

$$\|\Delta y\|^{p^- - 1}_{L^{p(x)}} \leq C \|f\|_{L^{p'(x)}},$$

hence the a priori bound (6) follows. $\square$

3.2 Optimal control problem

We will study the the existence of OCs (υ, y). First, we define the admissible control set Ξ as:

$$\Xi = \{(\upsilon, y) \mid (\upsilon, y) \in \mathfrak{A}_{ad} \times X, \text{such that } (\upsilon, y) \text{ satisfies the state equation (2)}\}.$$

Lemma 1. *The set Ξ of admissible pairs contains at least one element.*

Proof. By Theorem 5, for any $\upsilon \in \mathfrak{A}_{ad}$, there is a unique solution $y \in X$ to the state equation (2), since υ satisfies $0 < \alpha \leq \upsilon(x) \leq \beta$ and $f \in L^{p'(x)}(\Omega)$. Thus, for any $\upsilon \in \mathfrak{A}_{ad}$, there exists a corresponding y, so $\Xi \neq \emptyset$. $\square$

Theorem 6. *Let $p(x)$ satisfy (3), for $f \in L^{p'(x)}(\Omega)$ and $y_d \in L^{p(x)}(\Omega)$. Then, one can find an OC $\upsilon^* \in \mathfrak{A}_{ad}$ and an associated state $y^* \in X$ in a way that satisfies:*

$$\mathscr{J}(\upsilon^*, y^*) = \inf_{(\upsilon,y)\in\Xi} \mathscr{J}(\upsilon, y).$$

Proof. Given that $\mathscr{J}(\upsilon, y) \geq 0$, one can construct a sequence $\{(\upsilon_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ minimizing $\mathscr{J}$, with that:

$$\lim_{k\to\infty} \mathscr{J}(\upsilon_k, y_k) = \inf_{(\upsilon,y)\in\Xi} \mathscr{J}(\upsilon, y) < \infty.$$

Since $\upsilon_k \in \mathfrak{A}_{ad}$, we have $\alpha \leq \upsilon_k(x) \leq \beta$, so $\{\upsilon_k\}$ is bounded in $L^\infty(\Omega)$. Moreover, because $\mathscr{J}(\upsilon_k, y_k)$ is bounded, it follows that:

$$TV(\upsilon_k) \leq \mathscr{J}(\upsilon_k, y_k) \leq C,$$

where $C > 0$ a constant. Thus, $\{\upsilon_k\}$ is bounded in $BV(\Omega)$, by Theorem 4, we can extract a subsequence (also denoted by $\{\upsilon_k\}$) satisfying:

$$\upsilon_k \overset{*}{\rightharpoonup} \upsilon^* \text{ in } BV(\Omega), \quad \upsilon_k \to \upsilon^* \text{ strongly in } L^1(\Omega). \tag{8}$$

As $\mathfrak{A}_{ad}$ is closed and convex in $L^\infty(\Omega)$, and $\upsilon_k \to \upsilon^*$ in $L^1(\Omega)$, we get $\upsilon^* \in \mathfrak{A}_{ad}$.
Since $\{\upsilon_k\}$ is bounded in the space $L^\infty(\Omega)$, one can extract a subsequence (which we again denote by $\{\upsilon_k\}$) which converges strongly in $L^r(\Omega)$ for all $1 \le r < \infty$. In particular, this implies that

$$\upsilon_k \to \upsilon^* \text{ strongly in } L^{p'(x)}(\Omega) \text{ for } 1 \le p'(x) < \infty. \tag{9}$$

Now, by Theorem 5, each $y_k \in X$ satisfies (6). Since $f \in L^{p'(x)}(\Omega)$. Hence, $\{y_k\}$ remains bounded in the reflexive space X, ensures the existence of a a weakly convergent subsequence (still again denoted by $\{y_k\}$) and an element $y^* \in X$ satisfying:

$$y_k \rightharpoonup y^* \text{ weakly in } X. \tag{10}$$

Now, we need to show $(\upsilon^*, y^*) \in \Xi$. For each y_k, the weak form holds:

$$\int_\Omega \upsilon_k |\Delta y_k|^{p(x)-2} \Delta y_k \Delta\varphi \, dx = \int_\Omega f\varphi \, dx, \quad \forall \varphi \in X.$$

By (10), we have $\Delta y_k \rightharpoonup \Delta y^*$ weakly in $L^{p(x)}(\Omega)$. Note that, for all $p(x)$ satisfies (3), the nonlinear operator $T(\xi) = |\xi|^{p(x)-2}\xi$ is continuous and bounded from $L^{p(x)}(\Omega)$ to $L^{p'(x)}(\Omega)$. Thus,

$$|\Delta y_k|^{p(x)-2}\Delta y_k \rightharpoonup |\Delta y^*|^{p(x)-2}\Delta y^* \text{ weakly in } L^{p'(x)}(\Omega).$$

Moreover, by (9), we get $\upsilon_k |\Delta y_k|^{p(x)-2}\Delta y_k \rightharpoonup \upsilon^* |\Delta y^*|^{p(x)-2}\Delta y^*$ in $L^1(\Omega)$, allowing us to pass to the limit:

$$\int_\Omega \upsilon^* |\Delta y^*|^{p(x)-2}\Delta y^* \Delta\varphi dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in X.$$

Since $y^* \in X$ and $\upsilon^* \in \mathfrak{A}_{ad}$, we have $(\upsilon^*, y^*) \in \Xi$.
It remains now to verify the lower semicontinuity of the objective functional $\mathscr{J}$, namely, $\mathscr{J}(\upsilon^*, y^*) \le \liminf_{k\to\infty} \mathscr{J}(\upsilon_k, y_k)$.
Note that, $y \mapsto \int_\Omega |y - y_d|^{p(x)} dx$ is continuous in $L^{p(x)}(\Omega)$. Since $y_k \rightharpoonup y^*$ in X, and $X \hookrightarrow L^{p(x)}(\Omega)$ is compact as $2 \le p(x) < \infty$), we have $y_k \to y^*$ in $L^{p(x)}(\Omega)$, and this imply that:

$$\int_\Omega |y_k - y_d|^{p(x)} dx \to \int_\Omega |y^* - y_d|^{p(x)} dx, \quad \text{as } k \to \infty.$$

By (8), and Theorem 3, then TV is lower semicontinuous. Thus:

$$\begin{aligned}\mathscr{J}(\upsilon^*, y^*) = \int_\Omega |y^* - y_d|^{p(x)} dx + TV(\upsilon^*) &\le \lim_{k\to\infty} \int_\Omega |y_k - y_d|^{p(x)} dx + \liminf_{k\to\infty} TV(\upsilon_k)\\ &= \liminf_{k\to\infty} \mathscr{J}(\upsilon_k, y_k).\end{aligned}$$

Since $\{(\upsilon_k, y_k)\}$ is a minimizing sequence, $\mathscr{J}(\upsilon^*, y^*) \le \inf_{(\upsilon,y)\in\Xi} \mathscr{J}(\upsilon, y)$, and since $(\upsilon^*, y^*) \in \Xi$, it follows that (υ^*, y^*) is a minimizer. Therefore, the proof is complete. □

4 Relaxed optimal control problem

The nonlinear $p(x)$-biharmonic equation associated with the original control problem may exhibit degeneracy, in particular when $|\Delta y|$ approaches zero or when the control coefficient υ becomes small. Such degeneracy leads to a loss of uniform ellipticity and prevents the direct application of standard existence, uniqueness, and numerical approximation techniques. To address this difficulty, we introduce a relaxation strategy that restores strict ellipticity while preserving the variational structure of the problem. More precisely, to prevent degeneracy of the operator in regions where $|\Delta y|$ is small, we consider the following ε-regularized state equation:

$$\begin{cases} \Delta\left(\upsilon\, k_{\varepsilon,p}(|\Delta y|^2)\Delta y\right) = f, & \text{in } \Omega,\\ y = \dfrac{\partial y}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where the potential $k_{\varepsilon,p}$ is defined by

$$k_{\varepsilon,p}(r) = (\varepsilon + r)^{\frac{p(x)-2}{2}} + \varepsilon,$$

with $\varepsilon > 0$ a small parameter ensuring that the associated differential operator remains uniformly and strictly elliptic for all admissible states and controls.

This regularization allows us to rigorously analyze the relaxed problem in a Hilbert setting and to establish existence, uniqueness, and stability results for the state equation. It also provides a mathematically consistent framework for numerical approximation. We emphasize that allowing fully degenerate or vanishing controls without regularization would require alternative analytical tools, such as degenerate elliptic theory or measure-valued control formulations, which are beyond the scope of the present work and constitute a natural direction for future research. We then work in the space

$$H_0^2(\Omega) = \left\{ y \in H^2(\Omega) : y = \frac{\partial y}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

which is a Hilbert space endowed with the norm $\|y\|_{H_0^2(\Omega)} = \|\Delta y\|_{L^2(\Omega)}$, equivalent to the standard H^2-norm due to the imposed boundary conditions.

Proposition 2. *Let $\upsilon \in \mathfrak{A}_{ad}$. Define the operator $A_\varepsilon : H_0^2(\Omega) \to H^{-2}(\Omega)$ by:*

$$\langle A_\varepsilon y, \varphi\rangle = \int_\Omega u(x)\, k_{\varepsilon,p}(|\Delta y|^2)\Delta y \Delta \varphi \, dx, \quad \textit{for all } y, \varphi \in H_0^2(\Omega),$$

is bounded such that $\|A_\varepsilon\| \le \beta\sqrt{2\varepsilon^2 + 2^{p^+-1}\varepsilon^{p^+-2} + 2^{p^+-1}|\Omega|^{2-p^+}}$, strict monotone and coercive.

Proof. We start by the boundedness. For $\upsilon \in \mathfrak{A}_{ad}$, we have

$$\begin{aligned}
\|A_\varepsilon\| &= \sup_{\substack{y\in H_0^2(\Omega)\\ \|y\|_{H_0^2}\le 1}} \|A_\varepsilon y\|_{H^{-2}} = \sup_{\substack{y\in H_0^2(\Omega)\\ \|y\|_{H_0^2}\le 1}} \sup_{\substack{\varphi\in H_0^2(\Omega)\\ \|\varphi\|_{H_0^2}\le 1}} \langle A_\varepsilon y, \varphi\rangle_{H_0^2, H^{-2}} \\
&= \sup_{\substack{y\in H_0^2(\Omega)\\ \|y\|_{H_0^2}\le 1}} \sup_{\substack{\varphi\in H_0^2(\Omega)\\ \|\varphi\|_{H_0^2}\le 1}} \int_\Omega u(x)\, k_{\varepsilon,p}(|\Delta y|^2)\Delta y \Delta\varphi \, dx
\end{aligned}$$

Since $u \in \mathfrak{A}_{ad}$:

$$|\langle A_\varepsilon y, \varphi\rangle| \le \beta \int_\Omega |k_{\varepsilon,p}(|\Delta y|^2)\Delta y||\Delta\varphi| \, dx.$$

Substitute $k_{\varepsilon,p}(r) = (\varepsilon + r)^{\frac{p(x)-2}{2}} + \varepsilon$, we get:

$$|\langle A_\varepsilon y, \varphi\rangle| \le \beta \int_\Omega \left[(\varepsilon + |\Delta y|^2)^{\frac{p(x)-2}{2}} |\Delta y| + \varepsilon|\Delta y| \right] |\Delta\varphi| \, dx.$$

Apply Hölder's inequality we have:

$$|\langle A_\varepsilon y, \varphi\rangle| \le \beta \|\Delta\varphi\|_{L^2(\Omega)} \left\| (\varepsilon + |\Delta y|^2)^{\frac{p(x)-2}{2}} |\Delta y| + \varepsilon|\Delta y| \right\|_{L^2(\Omega)}.$$

Since $\|\varphi\|_{H_0^2} = \|\Delta\varphi\|_{L^2(\Omega)} \le 1$, gives:

$$\|A_\varepsilon y\|_{H^{-2}} \le \beta \left\| (\varepsilon + |\Delta y|^2)^{\frac{p(x)-2}{2}} |\Delta y| + \varepsilon|\Delta y| \right\|_{L^2(\Omega)}.$$

Thus:

$$\|A_\varepsilon\| \le \beta \sup_{\substack{y\in H_0^2(\Omega)\\ \|y\|_{H_0^2}\le 1}} \left(\int_\Omega \left[(\varepsilon + |\Delta y|^2)^{\frac{p(x)-2}{2}} |\Delta y| + \varepsilon|\Delta y| \right]^2 dx \right)^{1/2}.$$

Now, expand the square term, use the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ for $a = (\varepsilon+|\Delta y|^2)^{\frac{p(x)-2}{2}}|\Delta y|$, $b = \varepsilon|\Delta y|$ and integrating:

$$\|A_\varepsilon\| \leq \beta \sup_{\|y\|_{H_0^2}\leq 1} \left(2\int_\Omega (\varepsilon+|\Delta y|^2)^{p(x)-2}|\Delta y|^2\,dx + 2\varepsilon^2\|y\|^2_{H_0^2}\right)^{1/2}. \tag{12}$$

Since $p(x) \leq p^+ \geq 2$, $(\varepsilon+r)^{p(x)-2} \leq (\varepsilon+r)^{p^+-2} \leq 2^{p^+-2}(\varepsilon^{p^+-2}+r^{p^+-2})$, we have:

$$\int_\Omega (\varepsilon+|\Delta y|^2)^{p^+-2}|\Delta y|^2\,dx \leq 2^{p^+-2}\left(\varepsilon^{p^+-2}\int_\Omega |\Delta y|^2\,dx + \int_\Omega |\Delta y|^{2(p^+-1)}\,dx\right). \tag{13}$$

Again, by Hölder's inequality, we have

$$\int_\Omega |\Delta y|^{2(p^+-1)}\,dx \leq \left(\int_\Omega |\Delta y|^2\,dx\right)^{p^+-1}|\Omega|^{1-(p^+-1)} = \|\Delta y\|^{2(p^+-1)}_{L^2}|\Omega|^{2-p^+}. \tag{14}$$

Plugging (13) and (14) with $\|y\|_{H_0^2} \leq 1$ in (12), we get:

$$\|A_\varepsilon\| \leq \beta\sqrt{2\left(\varepsilon^2+2^{p^+-2}\varepsilon^{p^+-2}+2^{p^+-2}|\Omega|^{2-p^+}\right)}.$$

Proving the strict monotonicity of A_ε. We can easy see $\forall a,b\in\mathbb{R}$ and $p\geq 2$, the algebraic inequality:

$$\left[(\varepsilon+|a|^2)^{\frac{p(x)-2}{2}}a-(\varepsilon+|b|^2)^{\frac{p(x)-2}{2}}b\right]\cdot(a-b) \geq \varepsilon^{\frac{p(x)-2}{2}}|a-b|^2. \tag{15}$$

Let $y,z\in H_0^2(\Omega)$ and $\upsilon\in\mathfrak{A}_{ad}$

$$\begin{aligned}\langle A_\varepsilon y - A_\varepsilon z, y-z\rangle &= \int_\Omega \upsilon(x)\left[k_{\varepsilon,p}(|\Delta y|^2)\Delta y - k_{\varepsilon,p}(|\Delta z|^2)\Delta z\right]\Delta(y-z)\,dx\\ &\geq \alpha\int_\Omega \left[k_{\varepsilon,p}(|\Delta y|^2)\Delta y - k_{\varepsilon,p}(|\Delta z|^2)\Delta z\right](\Delta y-\Delta z)\,dx\\ &= \alpha\int_\Omega\left[\left((\varepsilon+|\Delta y|^2)^{\frac{p(x)-2}{2}}\Delta y-(\varepsilon+|\Delta z|^2)^{\frac{p(x)-2}{2}}\Delta z\right)+\varepsilon(\Delta y-\Delta z)\right](\Delta y-\Delta z)\,dx\end{aligned}$$

Now, apply (15) for $a=\Delta y$, $b=\Delta z$ and $p=p(x)$ we get

$$\begin{aligned}\langle A_\varepsilon y - A_\varepsilon z, y-z\rangle &\geq \alpha\int_\Omega\left(\varepsilon^{\frac{p(x)-2}{2}}+\varepsilon\right)(\Delta y-\Delta z)^2\,dx\\ &\geq \alpha\varepsilon\|y-z\|^2_{H_0^2(\Omega)}.\end{aligned}$$

Then A_ε is strict monotone.
Finish with the coercivity. Since $k_{\varepsilon,p}(r)\geq\varepsilon$, for all $y\in H_0^2(\Omega)$:

$$\langle A_\varepsilon y, y\rangle = \int_\Omega \upsilon(x)k_{\varepsilon,p}(|\Delta y|^2)(\Delta y)^2\,dx \geq \alpha\varepsilon\int_\Omega(\Delta y)^2\,dx = \alpha\varepsilon\|y\|^2_{H_0^2}.$$

Thus

$$\frac{\langle A_\varepsilon y,y\rangle}{\|y\|_{H_0^2}} \to \infty \quad \text{as } \|y\|_{H_0^2}\to\infty.$$

Hence, A_ε is coercive. □

We are now ready to address and see the well possedness of the relaxed OCP.

Theorem 7. *Let $f\in L^{p'(x)}(\Omega)$ and $u\in\mathfrak{A}_{ad}$. For each $\varepsilon>0$, the regularized state equation (11) admits a single weak solution $y_\varepsilon\in H_0^2(\Omega)$ satisfying:*

$$\int_\Omega \upsilon\, k_{\varepsilon,p}(|\Delta y_\varepsilon|^2)\Delta y_\varepsilon\Delta\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\varphi\in H_0^2(\Omega).$$

Proof. For fixed $y, z, \varphi \in H_0^2(\Omega)$, we need to prove that , the map:

$$t \mapsto \langle A_\varepsilon(y+tz), \varphi\rangle = \int_\Omega \upsilon(x) k_{\varepsilon,p}(|\Delta(y+tz)|^2)\Delta(y+tz)\Delta\varphi\, dx$$

is continuous. Since $\upsilon \in L^\infty(\Omega)$, $\Delta(y+tz) = \Delta y + t\Delta z$, and $k_{\varepsilon,p}(r) = (\varepsilon + r)^{\frac{p(x)-2}{2}} + \varepsilon$ is continuous in r, the integrand:

$$t \mapsto \upsilon(x)\left[(\varepsilon + |\Delta y + t\Delta z|^2)^{\frac{p(x)-2}{2}} + \varepsilon\right](\Delta y + t\Delta z)\Delta\varphi$$

is continuous in t. since $\Delta y, \Delta z, \Delta\varphi \in L^2(\Omega)$, and $\upsilon, k_{\varepsilon,p}$ are bounded, the integrand is bounded by an L^1-function, so the Lebesgue dominated convergence theorem ensures continuity of the integral as a function of t.

By Proposition 2 and the hemicontinuity of A_ε in $H_0^2(\Omega)$, we apply the Browder-Minty theorem, one can find that the equation $A_\varepsilon y = f$, for all $f \in L^{p'(x)}$, admits a unique $y_\varepsilon \in H_0^2(\Omega)$. Moreover, we can see as Theorem 5:

$$\|y_\varepsilon\|_{H_0^2(\Omega)} \le \|f\|_{L^{p'(x)}(\Omega)}. \tag{16}$$

□

Let $\mathscr{F}_\varepsilon$ be the approximate energy functional associated with the approximate boundary equation (11), defined by

$$\mathscr{F}_\varepsilon(y_\varepsilon) = \int_\Omega \upsilon(x) K_{\varepsilon,p}(|\Delta y_\varepsilon|^2)\, dx + \int_\Omega f y_\varepsilon\, dx \tag{17}$$

where the function $K_{\varepsilon,p} \colon (0,\infty) \to \mathbb{R}$ is given by $K_{\varepsilon,p}(r) = \frac{1}{p(x)}(\varepsilon + r)^{p(x)/2} + \varepsilon r$ and $k_{\varepsilon,p}(r) = K'_{\varepsilon,p}(r)$. This functional is designed to approximate the original energy functional

$$\mathscr{F}(y) = \int_\Omega \frac{\upsilon(x)}{p(x)}|\Delta y|^{p(x)}\, dx + \int_\Omega f y\, dx,$$

which corresponds to the state equation (2). It easy to see those two energy function have exactly unique minimizer $y_\varepsilon \in H_0^2(\Omega)$ and $y \in X$, respectively which are exactly the unique solution fulfilling the equations (11) and (2), respectively.

In order to see that $\mathscr{F}_\varepsilon(y_\varepsilon) \to \mathscr{F}(y)$, and $y_\varepsilon \to y$ strongly in $H_0^2(\Omega)$, we use the concept of Γ-convergence [8,16]. Its tool is especially useful when dealing with degenerate operators, where classical convergence methods may fail. The main advantages is its capacity to ensure the convergence of associated minimizers. More precisely, we will see if $\mathscr{F}_\varepsilon$ Γ-converges to F, denoted by $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} F$, and if y_ε is a sequence of minimizers of $\mathscr{F}_\varepsilon$, then every limit point of $\{y_\varepsilon\}$ is a minimiser of F. Let us first recall the Γ-convergence principal.

Definition 1([8,16]). *Let X be a separable Banach space endowed with a topology τ, and let $\{H_\varepsilon\}_{\varepsilon>0}$ be a sequence of functionals $H_\varepsilon : X \to \overline{\mathbb{R}}$. We say that H_ε Γ-converges to a functional $H : X \to \overline{\mathbb{R}}$ with respect to the topology τ if, for every $x \in X$, the following hold:*

- ***(Liminf inequality)*** *For every sequence $x_\varepsilon \to x$ in (X,τ), we have $H(x) \le \liminf_{\varepsilon\to 0} H_h(x_\varepsilon)$.*
- ***(Limsup inequality)*** *There exists a sequence $x_\varepsilon \to x$ in (X,τ) such that $H(x) \ge \limsup_{\varepsilon\to 0} H_h(x_\varepsilon)$.*

Proposition 3. *Let $f \in L^{p'(x)}(\Omega)$, $\upsilon \in \mathfrak{A}_{ad}$ and $p(x)$ satisfy (3). For each $\varepsilon > 0$, the sequence y_ε converges in $H_0^2(\Omega)$ to the unique minimizer y of $\mathscr{F}$, with $\mathscr{F}_\varepsilon(y_\varepsilon) \to \mathscr{F}(y)$, and $y_\varepsilon \to y$ strongly in $H_0^2(\Omega)$.*

Proof. The functional $\mathscr{F}_\varepsilon(y)$ is coercive in $H_0^2(\Omega)$. Since $\upsilon(x) \ge \alpha > 0$, $p(x) \ge 2$, and $K_{\varepsilon,p}(r) \ge \varepsilon r$, we have:

$$\int_\Omega \upsilon(x) K_{\varepsilon,p}(|\Delta y|^2)\, dx \ge \alpha\varepsilon \int_\Omega |\Delta y|^2\, dx = \alpha\varepsilon\|y\|^2_{H_0^2}.$$

The linear term is bounded using $H_0^2(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ and for all $f \in L^{p'(x)}(\Omega)$ we obtain:

$$\left|\int_\Omega f y\, dx\right| \le \|f\|_{L^{p'(x)}}\|y\|_{L^{p(x)}} \le C\|y\|_{H_0^2}.$$

Thus, $\mathscr{F}_\varepsilon(y) \to \infty$ as $\|y\|_{H_0^2} \to \infty$, so y_ε is bounded in $H_0^2(\Omega)$. Hence, we can extract a subsequence (denoted by y_ε) such that

$$y_\varepsilon \rightharpoonup y \text{ weakly in } H_0^2(\Omega), \tag{18}$$

$$y_\varepsilon \to y \text{ strongly in } L^{p(x)}(\Omega). \tag{19}$$

By (18), we have $\Delta y_\varepsilon \rightharpoonup \Delta y$ in $L^2(\Omega)$, and up to extracting a subsequence, we find that $|\Delta y_\varepsilon|^2 \to |\Delta y|^2$ a.e. Since $K_{\varepsilon,p}(r) \to r^{p(x)/2}$, and $\frac{u}{p(x)} \geq \frac{\alpha}{p^+} > 0$, Fatou's lemma gives:

$$\liminf_{\varepsilon\to 0} \int_\Omega \upsilon(x) K_{\varepsilon,p}(|\Delta y_\varepsilon|^2)\, dx \geq \int_\Omega \frac{\upsilon(x)}{p(x)} |\Delta y|^{p(x)}\, dx.$$

For the linear term, by (19), and $f \in L^{p'(x)}(\Omega)$ we have:

$$\int_\Omega f y_\varepsilon\, dx \to \int_\Omega f y\, dx.$$

Thus, $\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(y_\varepsilon) \geq \mathscr{F}(y)$.
Next, in order to verify the limsup inequality, we define a recovery sequence y_ε by setting $y_\varepsilon = y$ for an arbitrary $y \in H_0^2(\Omega)$. Then:

$$\mathscr{F}_\varepsilon(y) = \int_\Omega \upsilon(x) \left[\frac{1}{p(x)} (\varepsilon + |\Delta y|^2)^{p(x)/2} + \varepsilon |\Delta y|^2 \right] dx + \int_\Omega f y\, dx.$$

As $\varepsilon \to 0$, the integrand converges pointwise to $\frac{\upsilon}{p(x)} |\Delta y|^{p(x)}$. Since $p \leq p^+ < \infty$, and $|\Delta y|^2 \in L^2(\Omega)$, dominated convergence applies, so:

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(y) = \mathscr{F}(y).$$

Thus, $\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(y_\varepsilon) \leq \mathscr{F}(y)$.
Since $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$, and $\mathscr{F}_\varepsilon$, $\mathscr{F}$ are strictly convexe then they have unique minimizers, $y_\varepsilon \to y$ weakly in $H_0^2(\Omega)$, and $\mathscr{F}_\varepsilon(y_\varepsilon) \to \mathscr{F}(y)$. For strong convergence in $H_0^2(\Omega)$, since $\mathscr{F}_\varepsilon(y_\varepsilon) \to \mathscr{F}(y)$, the norm $\|y_\varepsilon\|_{H_0^2} \to \|y\|_{H_0^2}$ (due to uniform convexity and energy convergence), implying $y_\varepsilon \to y$ strongly. □

Similarly to Theorem 6, we can establish the existence of a regularized OC result as follow.

Theorem 8. *Let $f \in L^{p'(x)}(\Omega)$ and let $p(x)$ satisfy (3). Then, there exists an OC $\upsilon_\varepsilon^* \in \mathfrak{A}_{ad}$ and an associated state $y_\varepsilon^* \in H_0^2(\Omega)$ satisfying:*

$$\mathscr{J}(\upsilon_\varepsilon^*, y_\varepsilon^*) = \inf_{(\upsilon,y)\in\Xi_\varepsilon} \mathscr{J}(\upsilon, y),$$

where $\Xi_\varepsilon = \{(\upsilon,y) \mid \upsilon \in \mathfrak{A}_{ad}, y \in H_0^2(\Omega), (\upsilon,y)$ satisfies the equation (11)$\}$.

Lemma 2. *For every $\varepsilon > 0$, let $\mathscr{J}_\varepsilon$ be the regularized OCP functional defined by:*

$$\mathscr{J}_\varepsilon(\upsilon, y) = \begin{cases} \mathscr{J}(\upsilon, y) & \text{if } (\upsilon, y) \in \Xi_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the relaxed OCP admits at least one optimal pair $(\upsilon_\varepsilon^*, y_\varepsilon^*) \in \Xi_\varepsilon$.

Proof. The proof relies on the direct method of the calculus of variations. Let $\{(\upsilon_\varepsilon^k, y_\varepsilon^k)\} \subset \Xi_\varepsilon$ be a minimizing sequence for $\mathscr{J}_\varepsilon$. Since $\upsilon_\varepsilon^k \in \mathfrak{A}_{ad}$, the uniform L^∞ bound together with the total variation term implies the existence of a $C > 0$ such that $\|\upsilon_\varepsilon^k\|_{BV} \leq C$. For each υ_ε^k, the ε-regularized state equation admits a unique solution $y_\varepsilon^k \in H_0^2(\Omega)$. Moreover, the strict ellipticity induced by the regularization yields uniform bounds in $H_0^2(\Omega)$. Hence, the functional $\mathscr{J}_\varepsilon$ is coercive on Ξ_ε. By the compactness of bounded sets in $BV(\Omega)$, there exists $\upsilon_\varepsilon^* \in BV(\Omega)$ and up a subsequence $\{\upsilon_\varepsilon^k\}$ such that

$$\upsilon_\varepsilon^k \to \upsilon_\varepsilon^* \quad \text{in } L^1(\Omega), \qquad \upsilon_\varepsilon^k \rightharpoonup^* \upsilon_\varepsilon^* \quad \text{in } BV(\Omega).$$

Since $\mathfrak{A}_{ad}$ is closed under weak* convergence, $\upsilon_\varepsilon^* \in \mathfrak{A}_{ad}$. Furthermore, there exists $y_\varepsilon^* \in H_0^2(\Omega)$ such that $y_\varepsilon^k \rightharpoonup y_\varepsilon^*$ weakly in $H_0^2(\Omega)$.

Using Theorem 3, the TV is weakly lower semicontinuous in $BV(\Omega)$, and $\int_\Omega |y_\varepsilon^k - y_d|^{p(x)}\,dx$ is convex hence it is weakly lower semicontinuous. Therefore,

$$\mathscr{J}_\varepsilon(\upsilon_\varepsilon^*, y_\varepsilon^*) \leq \liminf_{k\to\infty} \mathscr{J}_\varepsilon(\upsilon_\varepsilon^k, y_\varepsilon^k).$$

Consequently, the limit pair $(\upsilon_\varepsilon^*, y_\varepsilon^*)$ belongs to Ξ_ε and minimizes the relaxed functional $\mathscr{J}_\varepsilon$. □

Now, we apply the notion of Γ-convergence in the sense of Definition 1 to the sequence of relaxed optimal control functionals $\{\mathscr{J}_\varepsilon\}$. Let us define the product space

$$V := L^1(\Omega) \times H_0^2(\Omega),$$

which endowed with the topology τ given by strong convergence in $L^1(\Omega)$ for the control variable and weak convergence in $H_0^2(\Omega)$ for the state variable. That is,

$$(\upsilon_\varepsilon, y_\varepsilon) \xrightarrow{\tau} (\upsilon, y) \quad \Longleftrightarrow \quad \begin{cases} \upsilon_\varepsilon \to \upsilon \text{ strongly in } L^1(\Omega), \\ y_\varepsilon \rightharpoonup y \text{ weakly in } H_0^2(\Omega). \end{cases}$$

The choice of this topology is justified by compactness properties of the admissible set. Indeed, the admissible controls are uniformly bounded in $BV(\Omega)\cap L^\infty(\Omega)$, which ensures relative compactness in $L^1(\Omega)$. Moreover, for each admissible control, the ε-regularized state equation admits a unique solution bounded in $H_0^2(\Omega)$. Consequently, every sequence $\{(\upsilon_\varepsilon, y_\varepsilon)\} \subset \Xi_\varepsilon$ with uniformly bounded energy admits a subsequence converging in the topology τ.

Now, we will prove that the regulraized OCP converges to the original one.

Theorem 9. *Let $(\upsilon_\varepsilon^*, y_\varepsilon^*) \in \Xi_\varepsilon$ be a minimizer of the regularized OCP functional $\mathscr{J}_\varepsilon(\upsilon, y)$. Let $(\upsilon^*, y^*) \in \Xi$ be a minimizer of the original OCP functional:*

$$\mathscr{J}_0(\upsilon, y) = \begin{cases} \mathscr{J}(\upsilon, y) & \textit{if } (\upsilon, y) \in \Xi, \\ +\infty & \textit{otherwise}, \end{cases}$$

Then, as $\varepsilon \to 0$, there is a subsequence along which:

$$\mathscr{J}_\varepsilon(\upsilon_\varepsilon^*, y_\varepsilon^*) \to \mathscr{J}_0(\upsilon^*, y^*) \tag{20}$$

$$\upsilon_\varepsilon^* \to \upsilon^* \textit{ strongly in } L^1(\Omega) \textit{ and weakly* in } BV(\Omega) \tag{21}$$

$$y_\varepsilon^* \to y^* \textit{ strongly in } H_0^2(\Omega) \tag{22}$$

Proof. Since $\upsilon_\varepsilon \in \mathfrak{A}_{ad}$, then by the L^∞-bound and the total variation constraint $\|\upsilon_\varepsilon\|_{BV} \leq C$. Then, there is $\upsilon^* \in BV(\Omega)$ and up a subsequence $\{\upsilon_\varepsilon\}$ such that (21) holds true for any $\upsilon^* \in BV(\Omega)$. Since $\mathfrak{A}_{ad}$ is closed under weak* convergence, $\upsilon^* \in \mathfrak{A}_{ad}$.

From (16), we have, $\{y_\varepsilon\}$ is bounded in $H_0^2(\Omega)$. Hence, we consider an element y and a subsequence $\{y_\varepsilon\}$ that converges weakly to y in $H_0^2(\Omega)$. By Proposition 3, $y_\varepsilon \to y^*$ strongly in $H_0^2(\Omega)$, where y^* solves the original state equation (2) with control υ^*.

Now, we will show that $\mathscr{J}_\varepsilon \xrightarrow{\Gamma} \mathscr{J}_0$ in V.

Consider a sequence $(\upsilon_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$ such that $\upsilon_\varepsilon \to \upsilon$ in $L^1(\Omega)$, $y_\varepsilon \rightharpoonup y$ weakly in $H_0^2(\Omega)$. If $(\upsilon, y) \notin \Xi$, then $\mathscr{J}_0(\upsilon, y) = +\infty$, and the inequality holds trivially, as $\liminf \mathscr{J}_\varepsilon(\upsilon_\varepsilon, y_\varepsilon) \geq 0$. If $(\upsilon_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon$, we have $\mathscr{J}_\varepsilon(\upsilon_\varepsilon, y_\varepsilon) = \mathscr{J}(\upsilon_\varepsilon, y_\varepsilon) = \int_\Omega |y_\varepsilon - y_d|^{p(x)}\,dx + TV(\upsilon_\varepsilon)$, and y_ε satisfies the regularized state equation (11) (or equivalently the unique minimezer of $\mathscr{F}_\varepsilon$ (17))

Since $\upsilon_\varepsilon \to \upsilon$ in $L^1(\Omega)$, and $\mathfrak{A}_{ad} \subset L^\infty(\Omega)$, we have $\upsilon_\varepsilon \to \upsilon$ in $L^q(\Omega)$ for $1 \leq q < \infty$ (by boundedness). By Proposition 3, for fixed υ, the solution y_ε of the regularized state equation converges strongly to the solution y of the original state equation in $H_0^2(\Omega)$. Here, $\upsilon_\varepsilon \to \upsilon$, and

$$\mathscr{F}_\varepsilon(y_\varepsilon) \to \mathscr{F}(y)$$

Thus, $(\upsilon, y) \in \Xi$.

Since $y_\varepsilon \to y$ in $H_0^2(\Omega)$, and $H_0^2(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, hence $y_\varepsilon \to y$ strongly in $L^{p(x)}(\Omega)$. Since the functional $y \mapsto \int_\Omega |y - y_d|^{p(x)}\,dx$ is continuous in $L^{p(x)}(\Omega)$, thus:

$$\int_\Omega |y_\varepsilon - y_d|^{p(x)}\,dx \to \int_\Omega |y - y_d|^{p(x)}\,dx.$$

Since the TV is lower semicontinuous, therefore:

$$\mathscr{J}(\upsilon,y) \leq \liminf_{\varepsilon\to 0} \mathscr{J}(\upsilon_\varepsilon,y_\varepsilon) = \liminf_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\upsilon_\varepsilon,y_\varepsilon).$$

Since $(\upsilon,y)\in\Xi$, $\mathscr{J}_0(\upsilon,y)=\mathscr{J}(\upsilon,y)$, so the liminf inequality holds.
For any $(\upsilon,y)\in\Xi$, construct a sequence $(\upsilon_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon$ such that $\upsilon_\varepsilon\to\upsilon$ in $L^1(\Omega)$, $y_\varepsilon\to y$ in $H_0^2(\Omega)$, and $\mathscr{J}_0(\upsilon,y)\geq \limsup_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\upsilon_\varepsilon,y_\varepsilon)$. Set $\upsilon_\varepsilon=\upsilon$, since $\upsilon\in\mathfrak{A}_{ad}$ is admissible for all ε, and let $y_\varepsilon\in H_0^2(\Omega)$ be the unique weak solution to the regularized state equation (11) for control υ. By Proposition 3, for fixed $\upsilon\in\mathfrak{A}_{ad}$, the solution y_ε converges strongly to the solution y of the state equation (2) in $H_0^2(\Omega)$. Thus,

$$(\upsilon_\varepsilon,y_\varepsilon)=(\upsilon,y_\varepsilon)\in\Xi_\varepsilon$$
$$\upsilon_\varepsilon=\upsilon\to\upsilon \text{ strongly in } L^1(\Omega)$$
$$y_\varepsilon\to y \text{ strongly in } H_0^2(\Omega)$$

Since $y_\varepsilon\to y$ strongly in $H_0^2(\Omega)$ we can also find that: $\int_\Omega |y_\varepsilon-y_d|^{p(x)}\,dx\to\int_\Omega|y-y_d|^{p(x)}\,dx$. Therefore:

$$\lim_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\upsilon_\varepsilon,y_\varepsilon)=\lim_{\varepsilon\to 0}\mathscr{J}(\upsilon,y_\varepsilon)=\mathscr{J}(\upsilon,y)=\mathscr{J}_0(\upsilon,y).$$

Since $\lim\mathscr{J}_\varepsilon(\upsilon_\varepsilon,y_\varepsilon)=\mathscr{J}_0(\upsilon,y)$, we have:

$$\limsup_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\upsilon_\varepsilon,y_\varepsilon)\leq\mathscr{J}_0(\upsilon,y),$$

satisfying the limsup inequality.
Therefore,

$$\mathscr{J}_\varepsilon\xrightarrow{\Gamma}\mathscr{J}_0 \text{ in } V.$$

Since $\mathscr{J}_\varepsilon\xrightarrow{\Gamma}\mathscr{J}_0$ in V, we analyze the minimizers $(\upsilon_\varepsilon^*,y_\varepsilon^*)\in\Xi_\varepsilon$. Since $\upsilon_\varepsilon^*\in\mathfrak{A}_{ad}$ satisfy $0<\alpha\leq\upsilon_\varepsilon^*\leq\beta$ a.e. Hence, $\{\upsilon_\varepsilon^*\}$ is bounded in $BV(\Omega)$. Since $y_\varepsilon^*\in H_0^2(\Omega)$ satisfies (16), we can easy show y_ε^* is bounded, and since υ_ε^* is bounded. By the Banach-Alaoglu theorem, there is $\upsilon^*\in BV(\Omega)$ and up a subsequence (still denoted by $\{\upsilon_\varepsilon^*\}$) such that $\upsilon_\varepsilon\overset{*}{\rightharpoonup}\upsilon^*$ weakly* in $BV(\Omega)$ and $\upsilon_\varepsilon\to\upsilon^*$ strongly in $L^1(\Omega)$. Since $\mathfrak{A}_{ad}$ is closed under weak* convergence, $\upsilon^*\in\mathfrak{A}_{ad}$. The boundedness in $H_0^2(\Omega$ imply a subsequence $\{y_\varepsilon^*\}$ converging weakly in $H_0^2(\Omega)$.

By the propreties of Γ-convergence; the liminf inequality ensures that if $(\upsilon_\varepsilon^*,y_\varepsilon^*)\to(\upsilon^*,y^*)$, then:

$$\mathscr{J}_0(\upsilon^*,y^*)\leq\liminf_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\upsilon_\varepsilon^*,y_\varepsilon^*).$$

The recovery sequence guarantees that for any $(\upsilon,y)\in\Xi$, there exists a sequence with $\mathscr{J}_\varepsilon\to\mathscr{J}_0$, so (υ^*,y^*) is a minimizer of $\mathscr{J}_0$.
By Proposition 3, $y_\varepsilon\to y^*$ strongly in $H_0^2(\Omega)$, where y^* solves the original state equation (2) with control υ^*.
Since $\mathscr{J}_\varepsilon(\upsilon_\varepsilon^*,y_\varepsilon^*)\leq\mathscr{J}_\varepsilon(\upsilon,y_\varepsilon)$ for any $(\upsilon,y_\varepsilon)\in\Xi_\varepsilon$, and the recovery sequence gives $\mathscr{J}_\varepsilon(\upsilon,y_\varepsilon)\to\mathscr{J}_0(\upsilon,y)$, we have:

$$\limsup_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\upsilon_\varepsilon^*,y_\varepsilon^*)\leq\mathscr{J}_0(\upsilon,y).$$

Combined with the liminf inequality, if (υ^*,y^*) is a minimizer, then:

$$\mathscr{J}_\varepsilon(\upsilon_\varepsilon^*,y_\varepsilon^*)\to\mathscr{J}_0(\upsilon^*,y^*).$$

□

5 Numerical experiments

In this section we present the numerical implementation of the relaxed optimal control problem. We use a finite element method for the discretization of regularized optimal control problem (1)–(11). The nonlinear state equation is solved through a fixed–point strategy, where each linearized system corresponds to a coercive fourth–order elliptic problem [6]. The total variation regularization of the control is handled by primal–dual Chambolle-type update within each outer iteration [7,10,11].

5.1 Discretization

Let $\mathscr{T}_h$ be a conforming, shape-regular triangulation of the domain Ω into closed triangles K. For every element $K \in \mathscr{T}_h$, we denote by $\mathbb{P}_r(K)$ the space of polynomials on K of total degree at most r. The mesh size is defined by $h = \max_{K\in\mathscr{T}_h} h_K$, with $h_K = \mathrm{diam}(K)$. The state y belongs to $H_0^2(\Omega)$, which requires a C^1–conforming finite element discretization. We adopt the Argyris finite element [4], a quintic triangular element that is globally C^1 and widely used in the discretization of fourth–order elliptic problems. This yields the discrete state space

$$Y_h = \left\{ y_h \in C^1(\overline{\Omega}) \;:\; y_h|_K \in \mathbb{P}_5(K) \;\; \forall K \in \mathscr{T}_h,\;\; y_h = \partial_n y_h = 0 \text{ on } \partial\Omega \right\} \subset H_0^2(\Omega).$$

The control υ lies in $BV(\Omega) \cap L^\infty(\Omega)$. A natural discrete counterpart is the space of piecewise constant functions:

$$U_h = \left\{ \upsilon_h \in L^\infty(\Omega) : \; \upsilon_h|_K \in \mathbb{P}_0(K) \; \forall K \in \mathscr{T}_h \right\}.$$

The box constraints $\alpha \le \upsilon \le \beta$ are enforced elementwise. Since $\nabla\upsilon_h$ is not defined across element interfaces, the discrete total variation is defined in the standard way through the broken gradient, [12, **?**]

$$TV_h(\upsilon_h) = \sum_{K\in\mathscr{T}_h} \int_K |\nabla_h \upsilon_h|\, dx,$$

where ∇_h denotes the elementwise gradient and its adjoint div_h.

Given a control $\upsilon_h \in U_h$, the discrete state $y_h \in Y_h$ solves

$$\int_\Omega \upsilon_h(x)\, k_{\varepsilon,p}\big(|\Delta y_h(x)|^2\big)\, \Delta y_h(x)\, \Delta\varphi_h(x)\, dx = \int_\Omega f(x)\, \varphi_h(x)\, dx, \qquad \forall \varphi_h \in Y_h. \tag{23}$$

Equation (23) is nonlinear in y_h. To solve it, we employ a fixed-point linearization that preserves the monotone structure of the operator. Given $y_h^{(m)} \in Y_h$, the next iterate is defined as the unique solution $y_h^{(m+1)} \in Y_h$ of

$$\int_\Omega \upsilon_h\, k_{\varepsilon,p}\left(|\Delta y_h^{(m)}|^2\right) \Delta y_h^{(m+1)}\, \Delta\varphi_h\, dx = \int_\Omega f\, \varphi_h\, dx, \qquad \forall \varphi_h \in Y_h. \tag{24}$$

Since $\upsilon_h \ge \alpha > 0$ and $k_{\varepsilon,p} \ge \varepsilon$, the bilinear form in (24) is coercive and continuous on Y_h, thereby guaranteeing well-posedness of every fixed-point step [6].

After computing the discrete state $y_h^{(m+1)} := y_h^{(k+1)}$, we update the control by approximately decreasing the functional

$$\Upsilon_h(\upsilon_h) = \int_\Omega \upsilon_h\, k_{\varepsilon,p}\big(|\Delta y_h^{(k+1)}|^2\big)(\Delta y_h^{(k+1)})^2\, dx + TV_h(\upsilon_h), \qquad \alpha \le \upsilon_h \le \beta. \tag{25}$$

The first term corresponds to a fixed–point linearization of the coupling between the control and the nonlinear regularized state equation. The minimization of Υ_h is not performed exactly at each iteration, since this would be unnecessarily costly. Instead, and following the strategy commonly employed in TV-regularized inverse problems [7, 10, 11], we apply one primal–dual proximal step per outer iteration.

Let $\rho > 0$ and $\sigma > 0$ be primal and dual step sizes satisfying the usual primal–dual stability condition $\rho\,\sigma\,\|\nabla_h\|^2 < 1$. Let q_h be the dual variable associated with TV_h; in the discrete setting, q_h is a vector-valued field defined elementwise and constrained by $|q_h(x)| \le 1$ a.e.

Given $\upsilon_h^{(k)}$, we perform the following updates.

(1) **Unconstrained descent step:** The first step incorporates only the smooth part of (25). A forward (explicit) descent step gives the intermediate quantity

$$\widetilde{\upsilon}_h = \upsilon_h^{(k)} - \rho\, k_{\varepsilon,p}(|\Delta y_h^{(k+1)}|^2)(\Delta y_h^{(k+1)})^2.$$

This part handles only the first term of Υ_h, hence no dual variable appears here.

(2) **Dual ascent step:** The total variation term is nonsmooth, and its convex conjugate is expressed in terms of the dual variable q_h. The corresponding dual update is the standard Chambolle proximal operator [10]. We update

$$q_h^{(k+1)} = \frac{q_h^{(k)} + \sigma\,\nabla_h \widetilde{\upsilon}_h}{\max\left(1, \,|q_h^{(k)} + \sigma\,\nabla_h \widetilde{\upsilon}_h|\right)}.$$

This step depends only on the TV part of Υ_h.

(3) **Primal update and box projection:** The primal update incorporates the adjoint of the TV operator, namely the discrete divergence div_h:

$$\upsilon_h^{k+1} = \mathscr{P}_{[\alpha,\beta]}\big(\widetilde{\upsilon}_h - \rho\,\mathrm{div}_h\, q_h^{k+1}\big),$$

where $\mathscr{P}_{[\alpha,\beta]}$ denotes the elementwise projection onto the admissible interval.

We now combine the fixed point iteration for the state with the primal–dual update for the control. It yields the following algorithm Combining the Picard solver for the state with the primal–dual update for the control yields the Algorithm 1.

Algorithm 1 Fixed–Point Algorithm for the Regularized OCP

Require: Mesh $\mathscr{T}_h$, spaces Y_h and U_h, data f, y_d, bounds α, β, parameter $\varepsilon > 0$, tolerance tol > 0.

Choose initial $\upsilon_h^0 \in U_h$ with $\alpha \le \upsilon_h^0 \le \beta$.
Initialize q_h^0 to zero. Set $k \leftarrow 0$.
repeat
  State update: Fixed-point iteration
  Initialize $y_h^{(0)} = y_h^{(k)}$.
  repeat
    Compute $y_h^{(m+1)} \in Y_h$ from (24).
    $m \leftarrow m+1$.
  until $\|y_h^{(m)} - y_h^{(m-1)}\|_{H^2} \le$ tol
  Set $y_h^{(k+1)} = y_h^{(m)}$.
  Control update: One primal–dual iteration
  $\widetilde{\upsilon}_h^{(k+1)} = \upsilon_h^{(k)} - \rho\, k_{\varepsilon,p}(|\Delta y_h^{(k+1)}|^2)(\Delta y_h^{(k+1)})^2$.
  $q_h^{(k+1)} = \dfrac{q_h^{(k)} + \sigma\nabla_h \widetilde{\upsilon}_h^{(k+1)}}{\max\big(1, |q_h^{(k)} + \sigma\nabla_h \widetilde{\upsilon}_h^{(k+1)}|\big)}$.
  $\upsilon_h^{(k+1)} = \mathscr{P}_{[\alpha,\beta]}\big(\widetilde{\upsilon}_h^{(k+1)} - \rho\,\mathrm{div}_h\, q_h^{(k+1)}\big)$.
  $k \leftarrow k+1$.
until $\|\upsilon_h^{(k)} - \upsilon_h^{(k-1)}\|_{L^\infty} + \|y_h^k - y_h^{(k-1)}\|_{H^2} \le$ tol
Output: $(\upsilon_h^{(k)}, y_h^{(k)})$.

5.2 Example

Now, we illustrate the behaviour of the proposed algorithm. In this experiment, the stopping criterion for the state solution and control variable produced by Algorithm 1 is terminated once the relative difference between two successive iterates satisfies

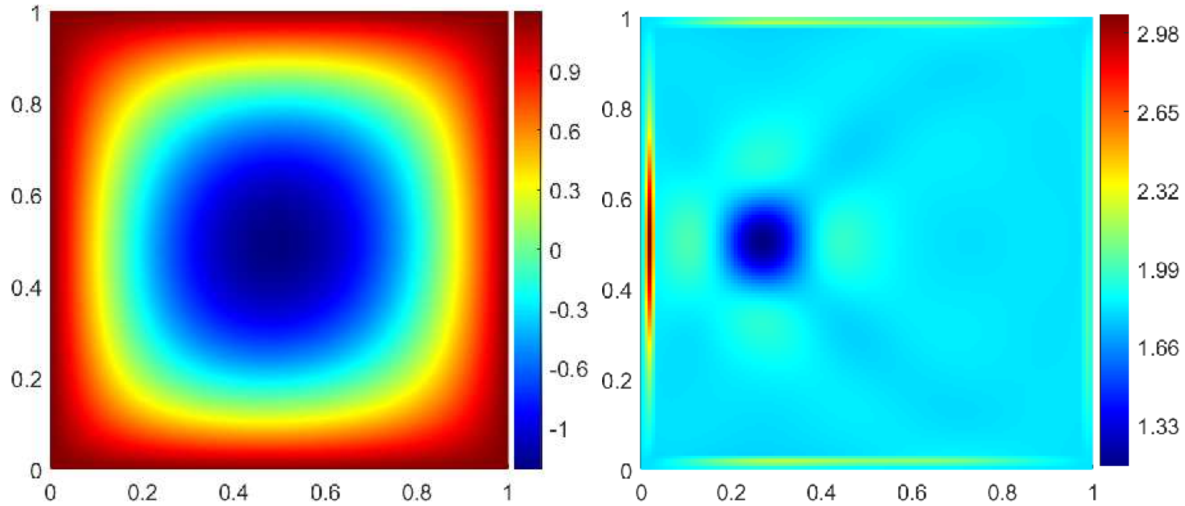
$$\frac{\|y_h^{(k+1)} - y_h^{(k)}\|_{H^2(\Omega)}}{\|y_h^{(k+1)}\|_{H^2(\Omega)}} \le 10^{-4} \text{ and } \frac{\|\upsilon_h^{(k+1)} - \upsilon_h^{(k)}\|_{L^2(\Omega)}}{\|\upsilon_h^{(k+1)}\|_{L^2(\Omega)}} \le 10^{-4}.$$

The regularization parameter is fixed to $\varepsilon = 10^{-3}$. We consider the square domain $\Omega = [0,1]^2$ and set the mesh size $h = 0.008$. The variable exponent $p(x)$ is evaluated at the mesh vertices and approximated by a piecewise constant function on each triangle. More precisely, for each element $K \in \mathscr{T}_h$ we set $p_h|_K = \frac{1}{3}\sum_{i=1}^{3} p(x_i)$, where x_1, x_2, x_3 are the vertices of K. In the numerical experiments we consider the exponent

$$p(x) = 2 + 0.5e^{-(x^2+y^2)},$$

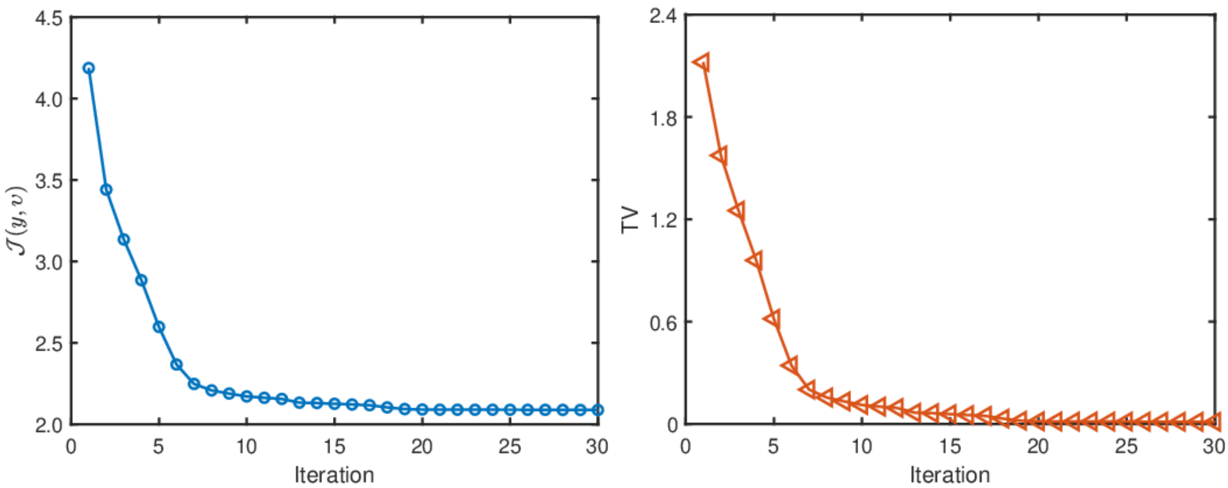
and use its elementwise average p_h in the computation of the nonlinear coefficient $k_{\varepsilon,p_h}(|\Delta y_h|^2)$. We also take the primal-dual parameters $\rho = 0.25$ and $\sigma = 0.25$, which satisfy the stability condition $\rho\sigma|\nabla_h|^2 < 1$ [11].

Figure 1 presents the numerical state y_h and the corresponding optimal control υ_h obtained by the proposed fixed-point and primal-dual scheme. The state is smooth across the domain and takes its maximum and minimum values in the interior, as expected for the biharmonic-type operator used in the model. The control function shows a piecewise-regular



(a) The state y_h. (b) The control υ_h.

Fig. 1: (a) Reconstructed discrete state y_h and (b) computed control υ_h obtained with the Algorithm 1.



(a) The cost function $\mathscr{J}$. (b) The total variation TV.

Fig. 2: (a) The objective $\mathscr{J}$ versus outer iterations and (b) the discrete TV seminorm of the control.

structure that is typical when total variation regularization is applied. The TV term encourages spatial flat regions and sharp but controlled transitions. This behaviour is consistent with the imposed box constraints $\alpha \leq \upsilon_h \leq \beta$.

Figure 2 shows the evolution of the cost functional $\mathscr{J}$ together with the discrete total variation of the control. The objective decreases very quickly during the first iterations and then approaches a plateau, which indicates that the state and control reach a consistent regime. The total variation also decreases at the beginning and stabilizes afterwards. This behaviour confirms that the primal-dual update correctly enforces TV smoothing while the fixed-point step adapts the control to the nonlinear diffusion governed by p_h. The monotone decay of both curves demonstrates the stability of the algorithm for the chosen parameter set.

Figure 3 displays the relative errors of the state and control in logarithmic scale. Both errors decrease rapidly, confirming that the fixed-point loop acts as a contraction for the discrete problem. The control error decreases slightly faster than the state error because the primal-dual update reacts immediately to the changes in the descent direction, while the state must be recomputed by solving a nonlinear elliptic problem that depends on p_h. After a few iterations the

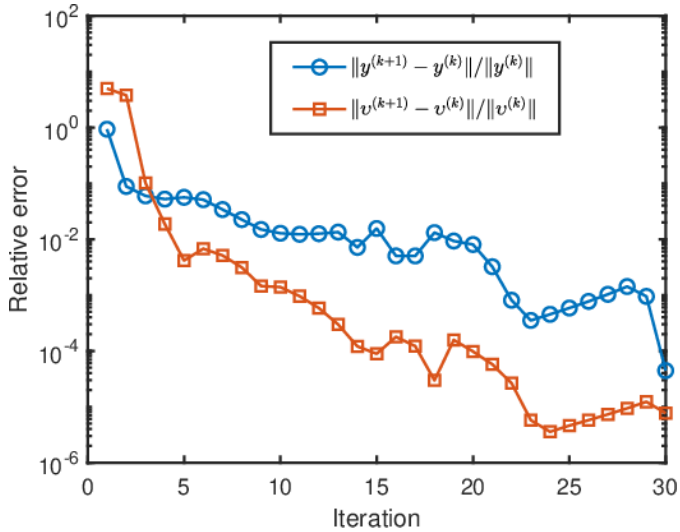


Fig. 3: Relative errors between successive outer iterates for state and control (log scale). Blue: relative error for the state $\|y_h^{k+1} - y_h^k\|/\|y_h^{k+1}\|$. Red: relative error for the control $\|\upsilon_h^{k+1} - \upsilon_h^k\|/\|\upsilon_h^{k+1}\|$.

two curves fall below the prescribed tolerance, and the algorithm stops. This behaviour shows that the method provides a stable and efficient numerical approximation of the regularized optimal control problem.

Conclusion

In this work, we studied an optimal control problem governed by a quasilinear $p(x)$-biharmonic equation with total variation-regularized coefficient control. Within a suitable variable-exponent framework, we established existence and uniqueness for the state equation and proved the existence of optimal controls over a bounded admissible set. To handle operator degeneracy, we introduced an ε-relaxed formulation that restores strict ellipticity. Using Γ-convergence, we showed that optimal solutions of the relaxed problems converge to those of the original as $\varepsilon \to 0$. Numerical experiments based on finite element discretization, fixed-point iteration, and a primal-dual algorithm illustrated the stability and convergence predicted by the theory. Future extensions may include weighted total variation or total generalized variation regularization.

Declarations

Competing interests: The author declares that there are no competing interests.
Funding: Not applicable
Availability of data and materials: No datasets were generated or analyzed.
Acknowledgments: The author thanks the anonymous referees for their valuable comments and suggestions, which helped improve the manuscript.

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