Variable-exponent Kirchhoff model for image restoration and thin-structures detection using the topological gradient method

Hamdi Houichet* — Anis Theljani* — Badreddine Rjaibi* — Maher Moakher*

* Université de Tunis El Manar, Ecole Nationale d'Ingénieurs de Tunis,
Laboratoire de Modélisation Mathématique et Numérique dans les Sciences de l'Ingénieur
B.P. 37, 1002 Tunis, Tunisia.
hamdi.houichet@gmail.com (H. Houichet), thaljanianis@gmail.com (A. Theljani),
badreddine.rjaibi@lamsin.rnu.tn (B. Rjaibi), maher.moakher@enit.rnu.tn (M. Moakher).

ABSTRACT. In this work, we propose a multiscale approach for a nonstandard higher-order based on the $p(\cdot)$-Kirchhoff operator. First, we consider a topological gradient approach for $p(\cdot) \equiv 2$ in order to detect thin structures. Then, we consider a nonlinear $p(\cdot)$-Kirchhoff equation where the variable exponent $1 < p(\cdot) \leq 2$ is chosen adaptively based on the map furnished by the topological gradient in order to preserve important features of the image. Finally, we present some numerical results to illustrate the effectiveness of our approach.

RÉSUMÉ. Dans ce travail, on propose une approche multiéchelle pour des équations aux dérivées partielles d'ordre supérieur, basée sur le modèle $p(\cdot)$-Kirchhoff pour résoudre les problèmes de détection des contours et la restoration des images. On considère le gradient topologique pour l'opérateur de bilaplace (i.e. $p(\cdot) \equiv 2$) afin de détecter les contours et les préserver. Ensuite, on présente une stratégie adaptative pour le choix du $p(\cdot)$. Ce choix est basé sur les structures fines détectées par le gradient topologique. On présente quelques résultats numériques pour illustrer l'efficacité de l'approche proposée.

KEYWORDS : $p(\cdot)$-Kirchhoff model, image denoising, biharmonic operator, topological gradient, speckle noise, nonlinear PDE.

MOTS-CLÉS : modèle de $p(\cdot)$-Kirchhoff, debruitage d'image, opérateur biharmonique, gradient topologique, bruit multiplicatif, EDP non-linéaire.

1. Introduction

Image restoration is a fundamental task in image processing that arises in widely diverse fields (geophysics, optics, medical imaging, etc). In this work, we are interested in the restoration of images highly corrupted with multiplicative noise. It is a challenging task in various fields and particularly in ultrasound medical imaging. The reason is that ultrasound images are strongly influenced by the quality of data usually corrupted with Rayleigh-distributed multiplicative noise so-called speckle noise [9, 10, 11] which usually affects image analysis methods by making important features hard to detect. We aim to reconstruct an image $u : \Omega \to \mathbb{R}$ from an observed one $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ which is degraded and contaminated by noise. The degradation model that we consider is the following:

$$f = u + \sqrt{u}\eta, \tag{1}$$

where $\eta : \Omega \to \mathbb{R}$ is a positive function and follows the Rayleigh-distribution. The reconstruction problem based on model (1) is an ill-posed inverse problem and thus regularization techniques are needed to overcome ill-posedness. Generally, the regularization technique turns the reconstruction problem based on model (1) into a well-posed optimization one where the energy to be minimized is the sum of a regularization term (mostly a semi-norm of a functional space fixed a priori) and a data fitting term. The main issue is how to choose the "best" regularization term which can selectively smooth a noisy image without losing significant features such as points and filaments. In this paper, we chose to minimize the following energy:

$$E_{p(x)}(u) := \int_\Omega \tfrac{1}{p(x)} |\Delta u|^{p(x)}\, \mathrm{d}x + \lambda \int_\Omega \frac{(f-u)^2}{u}\, \mathrm{d}x, \tag{2}$$

where $p(\cdot)$ is a measurable positive continuous function satisfying $p(x) \in]1, 2]$. The first part of the energy $E_{p(\cdot)}(\cdot)$ is the regularization term, the second one is the data fitting term and $\lambda > 0$ is weighting parameter. The smoothness of the solution is driven by the variable exponent $p(\cdot)$ which will be chosen adaptively in order to slow diffusion near thin-structures in order to sharpen and highlight them, and enhance diffusion in homogeneous regions. The restoration task is carried out in two steps: In an initial step, we use the topological gradient method for the biharmonic cost function (i.e. for $p(x) \equiv 2$), to detect important features. In a second step, we use the information furnished by the topological gradient (calculated for $p(x) \equiv 2$) in order to vary the exponent $p(\cdot)$ in the restoration process by considering a $p(\cdot)$-biharmonic model for $1 < p(x) \leq 2$.

This paper is organized as follows. In Section 2, we recall some basic notations and useful results of the generalized Lebesgue/Sobolev spaces. Then, we prove by standard variational techniques the existence of a minimizer of the energy (2). Section 3 is devoted to computing of the topological gradient for $p(\cdot) \equiv 2$. In section 4, we present the adaptive algorithm and the numerical scheme used to solve our problem. Then, we treat some numerical examples to test its efficiency and robustness.

2. Mathematical formulation of the problem

Preliminaries and general notations

For a bounded Lipschitz open set $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary, we assume that the variable-exponent function $p(\cdot) \in C(\overline{\Omega})$ satisfies the following condition:

$$1 < p^- := \inf_{x\in\overline{\Omega}} \leq p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) \leq 2.$$

We denote by $L^{p(x)}(\Omega)$ and $W^{2,p(x)}(\Omega)$ the variable-exponent generalized Lebesgue and Sobolev spaces, respectively. We denote by $|\cdot|_{p(x)}$ the norm of $L^{p(\cdot)}(\Omega)$ and by $\|\cdot\|_{2,p(x)}$ the norm of $W^{2,p(x)}(\Omega)$ where

$$|u|_{p(x)} = \inf\left\{\alpha > 0 : \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{p(x)} \mathrm{d}x \leq 1\right\}, \quad \text{and} \quad \|u\|_{2,p(x)} = \sum_{|\alpha|\leq 2} |D^\alpha u|_{p(x)}.$$

Both $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ and $(W^{2,p(\cdot)}(\Omega), ||\cdot||_{p(\cdot)})$ are separable and reflexive Banach spaces [6]. We introduce our working space which is defined as follows:

$$\mathcal{H}^{p(\cdot)}(\Omega) = W_0^{1,p(\cdot)}(\Omega) \cap W^{2,p(\cdot)}(\Omega),$$

which is also a separable and reflexive Banach space [12] for the norm $||\cdot|| = |\Delta\cdot|_{p(x)}$.

Existence and uniqueness of solution

In the following proposition, we establish the well-posedness of the minimization problem (2).

Proposition 1. *Let* $f \in W^{2,p(\cdot)}(\Omega) \cap L^2(\Omega)$ *with* $\inf_\Omega f > 0$*, then, the energy* (2) *admits a unique minimizer* u *in* $\mathcal{H}^{p(\cdot)}(\Omega)$ *satisfying*

$$\inf_\Omega f \leq u \leq \sup_\Omega f.$$

Proof. The existence and uniqueness follow from the classical compactness, the semi-continuity and the strict convexity arguments of the energy $E_{p(\cdot)}(\cdot)$. □

It is also standard that the minimizer of $E_{p(\cdot)}(\cdot)$ is the unique weak solution of the problem:

$$\begin{cases} \Delta^2_{p(x)} u + D_u\Phi(u, f) = 0, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(\cdot)$-biharmonic operator and $D_s\Phi(s,t) = \frac{s^2-t^2}{s^2}$ is the first derivative of $\Phi(s,t) = \frac{(t-s)^2}{s}$ with respect to $s \in \mathbb{R}^+_*$.

3. The topological gradient method

As mentioned in the introduction, we consider the topological gradient for the $p(\cdot)$-biharmonic operator only for the particular case $p(x) \equiv 2$. In what follows, we recall the basic idea of the topological gradient method. The starting point of this method is to insert a small crack $\sigma_\varepsilon := \{x_0 + \varepsilon\sigma(n)\}$ in the domain Ω, where $x_0 \in \Omega$, $\varepsilon > 0$, $\sigma(n)$ is a straight crack, and n is a unit vector normal to the crack and to minimize the energy norm outside the important features:

$$j(\Omega \setminus \sigma_\varepsilon) := J_2(u_\varepsilon) = \frac{1}{2}\int_{\Omega\setminus\sigma_\varepsilon} |\Delta u_\varepsilon|^2 \,\mathrm{d}x,$$

where u_ε is the unique solution of the following problem defined on the *perturbed domain* $\Omega_\varepsilon \overset{\text{def}}{=} \Omega \setminus \overline{\sigma}_\varepsilon$:

$$\begin{cases} \Delta^2 u_\varepsilon + D_u\Phi(u_\varepsilon, f) = 0, & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = \Delta u_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4}$$

where $\Delta^2 := \Delta\Delta$ is the bilaplace operator and $\partial\Omega_\varepsilon = \partial\Omega \cup \sigma_\varepsilon$.

Measuring the impact of such a modification of the domain on this cost function, the topological sensitivity method provides an asymptotic expansion of $j(\Omega_\varepsilon) - j(\Omega)$ as ε goes to zero. Generally, this asymptotic expansion can be written as follows [1, 2, 3, 8]:

$$J_2(u_\varepsilon) - J_2(u_0) = \rho^2 G(x_0, n) + o(\rho^2),$$

where the topological gradient is

$$G(x_0, n) = -\frac{2\pi}{3}\Delta u(x_0)(n,n)\Delta v(x_0)(n,n), \tag{5}$$

and v is the solution of the following adjoint problem: *Find* $v \in \mathcal{H}^2(\Omega)$ *such that*

$$\begin{cases} \Delta^2 v + D_u\Phi^2(u, f)v = -\Delta^2 u, & \text{in } \Omega, \\ v = \Delta v = 0, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Therefore, the topological gradient (5) can be written as [4]:

$$G(x_0,\theta) = -\frac{2\pi}{3}P\Big(\partial_{xx}u_0(x_0), \partial_{yy}u_0(x_0), \partial_{xy}u_0(x_0), \theta\Big)P\Big(\partial_{xx}v_0(x_0), \partial_{yy}v_0(x_0), \partial_{xy}v_0(x_0), \theta\Big),$$

where $P : \mathbb{R}^4 \to \mathbb{R}$ is the π-periodic function with respect to θ given by:

$$P(\alpha, \beta, \gamma, \theta) = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta)\cos(2\theta) + \gamma \sin(2\theta).$$

4. Implementation

The smoothness of solution of the minimization problem (2) is driven by the variable exponent $p(x)$. The selection of $p(\cdot)$ is performed with the help of the topological gradient which has been used as an efficient edge- and thin structure-detector. This particularity makes it well suited to control and locally select the exponent using the following algorithm:

Algorithm 1 MAIN ALGORITHM

Given f and λ.

1) For $p(\cdot) \equiv 2$, compute u and v which solve equations (4) and (6), respectively.
2) Compute the topological gradient $G(x_0, n)$ for each point $x_0 \in \Omega$.
3) Update $p(\cdot)$ to obtain a new exponent $p_a(\cdot)$ and solve (3) for $p(\cdot) \equiv p_a(\cdot)$.

In order to update to exponent $p(\cdot)$, we use the following formula:

$$p_a(x) = 1 + \exp(-\kappa |G(x, n)|), \ \forall x \in \Omega,$$

where $\kappa > 0$ is a constant. In homogeneous regions, we have $G(x, n) \approx 0$ leading to a new exponent $p_a(x)$ close to 2. Then, the model behaves like the biharmonic equation leading to strong diffusion and hence noise is damped. Near thin-structures, $G(x, n) \approx 0$ is very important and therefore $p_a(x)$ will be close to 1 which slows down diffusion. However, for such a choice of $p_a(\cdot)$, equation (2) is strongly nonlinear. For that, we introduce an artificial time variable t and for any fixed number T, we transform our problem to the following time-dependent one:

$$\begin{cases} u_t = -\nabla E_{p(\cdot)}(u), & \text{in } \Omega \times (0, T], \\ u(., t = 0) = f, & \text{in } \Omega. \end{cases} \tag{7}$$

After that, we consider *the splitting convexity* method (see [5, 7]) to solve problem (7). The basic idea of this method is to split the functional $E_{p(\cdot)}$ into a convex part treated implicitly, and a concave one treated explicitly. In our case, we split the energy $E_{p(\cdot)}$ as follows:

$$E_{p(\cdot)} = E_{1,2} - E_{2,p},$$

where

$$\begin{cases} E_{1,2} = \dfrac{c_1}{2}\displaystyle\int_\Omega |\Delta u|^2 \, \mathrm{d}x + \dfrac{c_2}{2}\int_\Omega |u|^2 \, \mathrm{d}x, \\ E_{2,p} = -\displaystyle\int_\Omega \dfrac{1}{p(x)}|\Delta u|^{p(x)} \, \mathrm{d}x - \lambda \int_\Omega \Phi(u,f) \, \mathrm{d}x + \dfrac{c_1}{2}\int_\Omega |\Delta u|^2 \, \mathrm{d}x + \dfrac{c_2}{2}\int_\Omega |u|^2 \, \mathrm{d}x, \end{cases}$$

and c_1 and c_2 are two positive constants. Let τ be the time-step, and write $t_k = k\tau$, $u^k(x) = u(x, t_k)$, with $k = 1, 2, \ldots, [T/\tau] - 1$, then, at each time step, we solve:

$$\frac{u^{k+1} - u^k}{\tau} + c_1\Delta^2 u^{k+1} + c_2 u^{k+1} = \Delta^2_{p(x)} u^k + c_1 \Delta^2 u^k + c_2 u^k + \lambda D_u \Phi(u^k, f).$$

5. Numerical experiments

In the following, we present some numerical results to illustrate the performance of the proposed approach for the speckle noise removal and thin-structures detection. In Figure 1, we consider a synthetic and blood cells images which are corrupted by speckle noise with variance $\sigma^2 = 0.02$. The images illustrated below show the efficiency of the topological gradient method for the detection of edges and/or thin structure and the nonlinear $p(\cdot)$-Kirchhoff model for the denoising task. The main difference between the noisy image (see Figures 1(a) and 1(d)) and restored one (see Figures 1(c) and 1(f)) is compared quantitatively by using the SNR and SSIM indicators.

6. Conclusion

In this paper, we have presented a new approach to restore images corrupted by speckle multiplicative noise. The proposed approach combines the advantages of the topological gradient method for thin-structures detection and the anisotropic diffusion model based on the $p(\cdot)$-Kirchhoff operator for the denoising. The numerical experiments show the good quality in the recovering of the thin structures as well as the image denoising.

7. References

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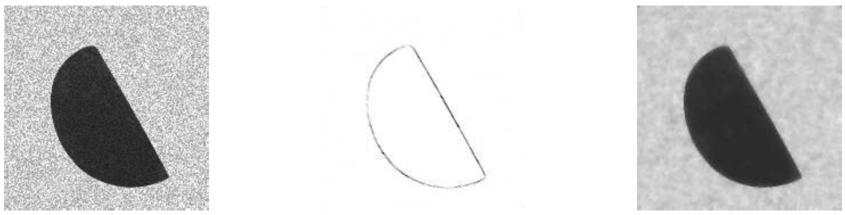

(a) SNR=3.72dB, SSIM= 0.26. (b) Edges of the image. (c) SNR=12.03dB, SSIM= 0.76.

(d) SNR=17.13dB, SSIM= 0.59. (e) Thin structures of the image. (f) SNR=21.02dB, SSIM= 0.8

Figure 1. *From left to right and top to bottom: Noisy image, detected important features and restored image.*

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