# Topological gradient approach to edge detection and speckle noise removal in ultrasound images

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## Introduction

It is well-known that the ultrasound images are corrupted by the speckle noise [2]. The observed image is assumed to be positive and decomposed as follows

$$g = u + \sqrt{u}\eta,$$

where the speckle noise $\eta$ is a positive function and follows the Rayleigh-distribution.

## Problem formulation

Let $\Omega$ be an open subset in $\mathbb{R}^2$ is the image domain.
To remove the speckle noise from an ultrasonic image, we study the following variational denoising model

$$\min_{\{u>0,\ u\in H^1(\Omega)\}} \left\{ \frac{\lambda}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega \frac{(g-u)^2}{u}\,dx \right\}, \tag{1}$$

where $\lambda > 0$ is a weighting factor between the regularization term and the data fidelity term. The associated Euler-Lagrange equation is given by

$$\begin{cases} -\mathrm{div}(\lambda\nabla u_0) + \dfrac{{u_0}^2 - g^2}{{u_0}^2} = 0, & \text{in } \Omega, \\ \partial_n u_0 = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

where $\partial_n$ denotes the normal derivative and $n$ is the outward unit normal to $\partial\Omega$.
**This equation admits a unique solution** $u_0$ **in** $H(\Omega) := \{u > 0;\ u \in H^1(\Omega)\}$.

**Edge detection:** Among several techniques proposed to **restore** the speckle noise arising in medical ultrasounic image, we resort, in this poster, to **the topological gradient method** [1].

## The topological sensitivity analysis

- For a small parameter $\rho > 0$, let $\Omega_\rho = \Omega \setminus E_\rho(x_0)$ be the perturbed domain obtained by removing an ellipse of size $\rho$ around the point $x_0$.
- The topological asymptotic expansion of $j(\Omega)$ when $\rho$ goes to zero is the following
$$j(\Omega_\rho) - j(\Omega) = \rho^2\mathcal{G}(x_0) + o(\rho^2),$$
where $\mathcal{G}(x_0)$ is called the topological gradient at $x_0$.
- To preserve the edges of an image and smooth it elsewhere, we compute the asymptotic expansion of the following cost function
$$j(\Omega_\rho) = J(u_\rho) = \frac{1}{2}\int_{\Omega_\rho} |\nabla u_\rho|^2\,dx, \tag{3}$$
where $u_\rho$ is the solution of the following perturbed equation
$$\begin{cases} -\mathrm{div}(\lambda\nabla u_\rho) + \dfrac{u_\rho^2 - g^2}{u_\rho^2} = 0, & \text{in } \Omega_\rho, \\ \partial_n u_\rho = 0, & \text{on } \partial\Omega_\rho. \end{cases} \tag{4}$$
- **The Topological gradient** $\mathcal{G}(x_0)$
$$\mathcal{G}(x_0) = -\pi\left(ab\left(1 - \frac{g^2(x_0)}{u_0^2(x_0)}\right)(v_0(x_0) - u_0(x_0)) + \lambda(a+b)\nabla u_0(x_0)\cdot\mathcal{P}\nabla v_0(x_0)\right),$$
where
$$\mathcal{P} = \begin{pmatrix} b\cos^2\varphi + a\sin^2\varphi & (a-b)\cos\varphi\sin\varphi \\ (a-b)\cos\varphi\sin\varphi & b\sin^2\varphi + a\cos^2\varphi \end{pmatrix},$$
for all $\varphi \in [0, 2\pi]$ and $a, b > 0$ are the lengths of the semi-axes of $E$.

## The adjoint problem

The function $v_0$ defined in the topological gradient formula is the solution of the following adjoint problem

$$\begin{cases} -\mathrm{div}(\lambda\nabla v_0) + \dfrac{2g^2}{{u_0}^3}v_0 = \dfrac{{u_0}^2 + g^2}{{u_0}^2}, & \text{on } \Omega, \\ \partial_n v_0 = 0, & \text{in } \partial\Omega. \end{cases} \tag{5}$$

**If we take $a = b = 1$, we find the topological gradient for a perforated domain by a ball.**
**When we take $a = 1$ and tend $b$ to zero, we find the topological gradient for a cracked domain.**

## Restoration algorithm

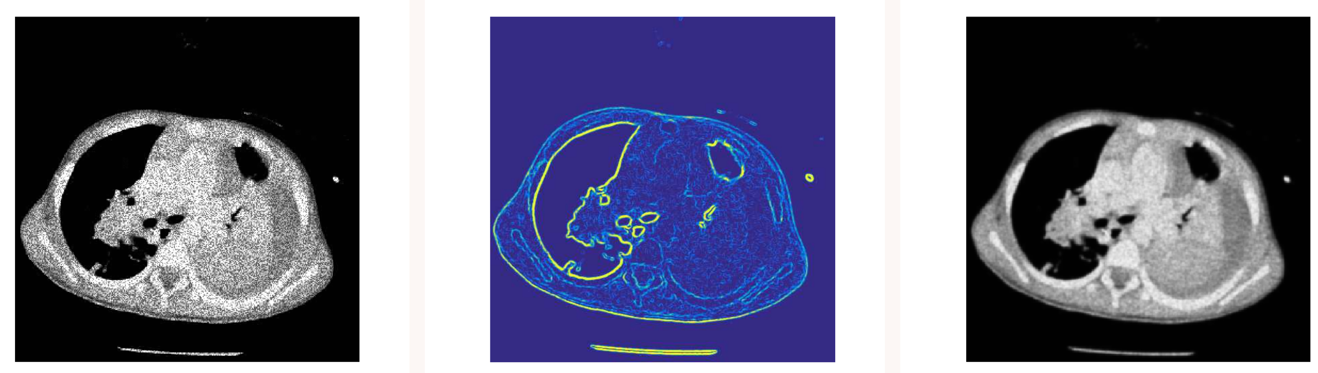
The steps of the algorithm are the following:

- Initialize: $\lambda = \lambda_0$.
- Compute $u_0$: the solution of equation (2).
- Compute $v$: the solution of equation (5).
- Compute the topological gradient $\mathcal{G}(x_0)$ for each point $x_0 \in \Omega$.
- Take
$$\lambda = \begin{cases} \lambda_1 & \text{if } x_0 \in \{x \in \Omega \text{ such that } \mathcal{G}(x_0) < \alpha < 0\}, \\ \lambda_0 & \text{otherwise } (\lambda_1 \ll \lambda_0). \end{cases}$$
- Compute $u_0$ with the new value of $\lambda$.

where $\alpha$ is a given negative threshold.

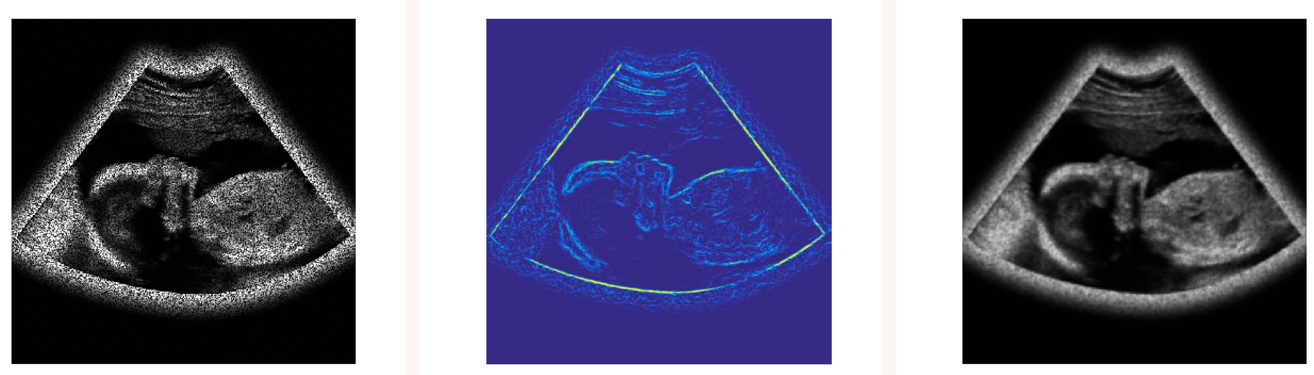
## Numerical examples

From left to right: Noisy image, contour and restored image, respectively.
**•Chest image**



snr=11.88dB, ssim=0.72 snr=18.74dB, ssim=0.88

The edges obtained by the topological gradient with threshold $\alpha = -200$ and the recovered image with $\lambda = 1.7$.
**•Sonography image**



snr=6.54. ssim=0.63 snr=16.01dB, ssim=0.87

## Conclusion and references

**Conclusion**

- We have studied a new method based on the topological sensitivity approach to remove speckle noise in ultrasound images.
- The edges are well preserved and the image is smoothed elsewhere during the restoration process.
- We can consider a similar approach for a higher-order PDE in Ultrasound Computed Tomography.

**References**

[1] A. Drogoul and G. Aubert, The topological gradient method for semi-linear problems and application to edge detection and noise removal, *Inverse Problem and Imaging*, **10** (2016), 51–86.
[2] Z. Jin and X. Yang, A variational model to remove the multiplicative noise in ultrasound images, *J. Math. Imaging Vis.*, **39** (2011), 62–74.