Split-convexity method for image restoration

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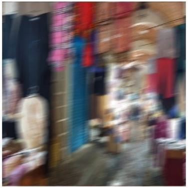
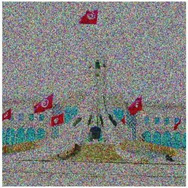
E N I T LAMSIN

Outline

What is image restoration problem

- Our story begins with a blurry and/or noisy image...





What is image restoration problem

- Our story begins with a blurry and/or noisy image...

- Our objective : Denoise the image ! ! !
- Different tools :
  - Statistics Estimators.
  - Scattered representations.
  - Inverse problems and regularization techniques.
  - Partial differential equations.

$\Rightarrow$ The last two approach are envisaged here.

Let Ω be a bounded and Lipschitz subset in $\mathbb{R}^2$. We begin with the following model [1]

$$f = Ru + (Ru)^m \eta, \tag{M}$$

where

- f is the observed image ($f : L^2(\Omega) \longrightarrow \mathbb{R}$ (or $\mathbb{R}^3$)).
- R is a linear blurring operator.
- η is the noise follow Gauss, Cauchy, Poisson or Gamma lows.
- u is the recovered image, and $m \in \mathbb{R}$.

Remark

In the model (M) :

- If $R = \mathrm{id}$ and $m = 0$, we have the additive model
$$f = u + \eta.$$
- If $R = \mathrm{id}$ and $m = 1/2$, we have the multiplicative model
$$f = u + \eta\sqrt{u}.$$

In this work, we focus on the ultrasound image which is contaminated by speckle noise

$$f = u + \eta\sqrt{u}. \tag{S}$$

1. T. Loupas, PhD thesis, University of Edinburgh, UK, 1988.

Regularization techniques

To reconstruct the image u from f, we shall solve the following minimization problem

$$\min_{u>0\ u\in X(\Omega)} \underbrace{\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, \mathrm{d}x}_{\textit{Regularizer}} + \lambda \underbrace{\int_\Omega \frac{(f-u)^2}{u}\, dx}_{\textit{Fidelity}} \tag{1}$$

where $X(\Omega)$ is a Banach space (as BV-space, L^2, $W^{k,p}$, etc.),

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Why the regularization term $J(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, \mathrm{d}x$?

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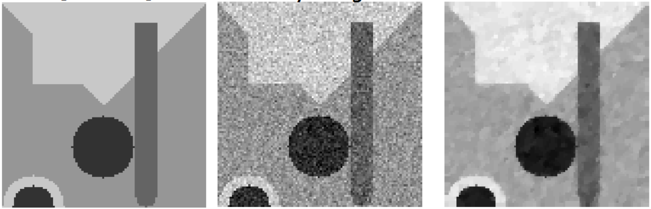
Why the regularization term $J(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx$?

If we consider

- **The Total Variation :**

$$TV(u) := \sup \left\{ \int_\Omega u(x) \mathrm{div}\phi(x) dx \; \phi \in \mathscr{C}_c^1(\Omega),\ \|\phi\|_\infty \leq 1 \right\},$$

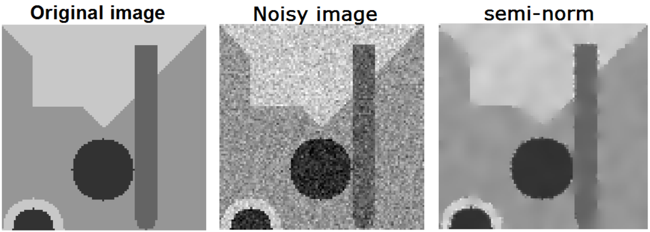
Original image Noisy image TV



- H^1-**seminorm :**

$$J(u) := \int_\Omega |\nabla u(x)|^2 \mathrm{d}x,$$

and $\mathcal{X}(\Omega) = W^{1,2}(\Omega)$.



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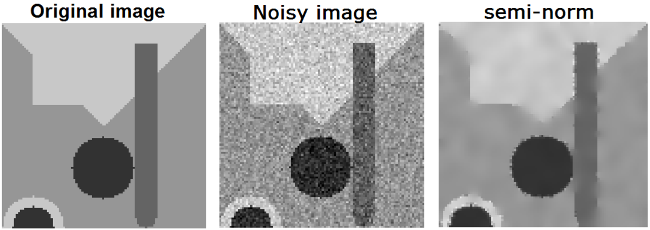

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$$\int_\Omega |Au|^1 dx \longleftrightarrow \int_\Omega |Au|^2 dx, \quad A = \Delta, \nabla, \cdots$$

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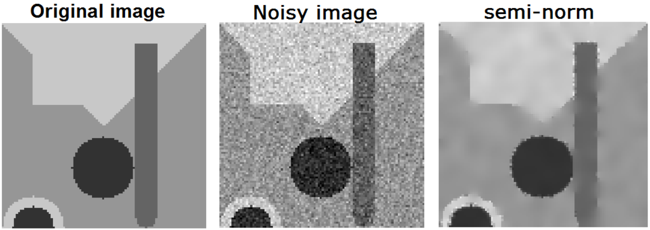

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$$\int_\Omega |Au|^{p(x)} dx$$

What is the difference between thin structures and contours ?

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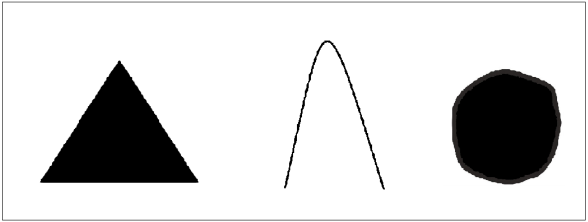
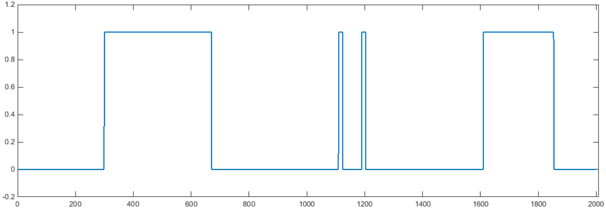
- Thin structures.
  - Objects of codimension ≥ 1, filaments (curves), points...
  - Discontinuities without jump of intensity.
  - The size of the structures depends on the resolution of the image.
- Edges
  - Objects of dimension 1.
  - Jumping intensity through the discontinuity.
  - Profile in the form of the Heaviside function.





Outline

Regularization method

We consider the following minimization problem

$$\min_{u>\mathbf{0},\, u\in X(\Omega)} E_{p(x)}(u), \tag{2}$$

where

$$E_{p(x)}(u) := J(u) + \lambda \int_\Omega \frac{(f-u)^{\mathbf{2}}}{u}\,\mathrm{d}x,$$

where $J(u) = \displaystyle\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx$, and $p:\ \Omega \to]1,2]$ continuous function is called exponent and satisfy

$$1 < p^- \leq p(x) \leq p^+ \leq 2$$

$$p^- = \inf_{x\in\Omega} p(x), \qquad p^+ = \sup_{x\in\Omega} p(x).$$

The working space is

$$X(\Omega) = \left\{ u \in W^{\mathbf{2},p(x)} : \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \right\}. \quad \text{(variable exponent space)}$$

The variable exponent Sobolev Spaces

- For measurable u define the variable-exponent Lebesgue space

$$L^{p(x)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\},$$

with Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\tau > 0 : \int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)}\mathrm{d}x \le 1\right\}.$$

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- The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\right\},$$

is equipped with the norm

$$\|u\|_{k,p(x)} := \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)}.$$

$\Rightarrow$ $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ **and** $(W^{k,p(x)}(\Omega), \|\cdot\|_{p(x)})$ **are separable, reflexive and convex Banach spaces**.

Existence and uniqueness :

Theorem

Let $f \in X(\Omega) \cap L^2(\Omega)$, then, the minimization problem (2) admits a unique minimizer u in $X(\Omega) \cap L^2(\Omega)$.

The solution u of the minimization problem (2) is a weak solution of the Euler-Lagrange equation :

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) + W'(u,f) = 0, & \text{in } \Omega, \\ \frac{\partial |\Delta u|^{p(x)-2}\Delta u}{\partial n} = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

where $W(u,f) = \lambda \frac{(f-u)^2}{u}$.

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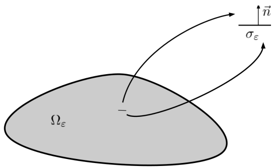
How to choose the variable exponent p ?

The topological gradient method : principal

We perturb the domain Ω by :

- Inserting a small crack : Let σ be a variety of dimension 1 containing the origin. Denote by $\sigma_\varepsilon = \varepsilon\sigma$ and $\Omega_\varepsilon = \Omega \setminus \overline{\{x_0 + \sigma_\varepsilon(\vec{n})\}}$.



- We minimize the cost function (with $p(x) = 2$) outside the important features :

$$J_\varepsilon(u) = \frac{\alpha}{2} \int_{\Omega_\varepsilon} |\Delta u|^2 \, \mathrm{d}x, \tag{4}$$

More generally, the variation of the cost function, has the following asymptotic expansion :

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^2 G(x_0, \vec{n}) + o(\varepsilon^2),$$

$G(x_0, \vec{n})$ is called the topological gradient at the point x_0

Remark

The points sought are the points associated with a large topological gradient in absolute value.

The topological gradient method : expression

We can deduce the expression of the topological gradient for a cracked domain.

$$G(x_0, \vec{n}) = -\alpha \frac{2\pi}{3} \nabla(\nabla u(x_0) \cdot \vec{n}) \cdot \vec{n} \nabla(\nabla v(x_0) \cdot \vec{n}) \cdot \vec{n}, \tag{5}$$

u is the solution of :

$$\begin{cases} \alpha \Delta^2 u + D_u W(u, f) = 0, & \text{in } \Omega, \\ \frac{\partial \Delta u}{\partial n} = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

v solves the following adjoint equation :

$$\begin{cases} \alpha \Delta^2 v + \lambda D_u^2 W(u, f) v = -\alpha \Delta^2 u, & \text{in } \Omega, \\ \frac{\partial \Delta v}{\partial n} = \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

$D_u^2 W(\cdot, f)$ is the second derivatives of $W(\cdot, f)$.

- $G(x_0, \vec{n})$ is the topological gradient for forth-order operator $\Rightarrow$. It is sensitive to important features detection.

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Algorithm 2 computation of G

Given f and λ.

1. For $p(\cdot) \equiv 2$, compute u and v which solve equations (6) and (7), respectively.
2. Compute the topological gradient $G(x_0, n)$ for each point $x_0 \in \Omega$.
3. Update $p(\cdot)$ and $q(\cdot)$ to obtain a new exponent $p_a(\cdot)$. Then, solve (3) for $p(\cdot) \equiv p_a(\cdot)$.

We use the following formula for the exponent $p(\cdot)$:

$$p(x) = 1 + e^{-\mu |G(x)|}, \text{ for all } x \in \Omega, \text{ and } \mu > 0.$$

- In homogeneous regions : $G(x) \approx 0 \Rightarrow p(x)$ closes to 2 (Strong diffusion).
- Near important features : $G(x)$ is very important $\Rightarrow p(x)$ closes to 1 (Slows down diffusion).

Outline

Question : How to solve, numerically, the nonlinear problem (3) ?

2. D. J. Eyre, An unconditionally stable one-step scheme for gradient systems, Unpublished article, 1998

3. K. Glasner and S. Orizaga, Journal Comp. Phy., 2016

Question : How to solve, numerically, the nonlinear problem (3) ?

We transform our problem to the time-dependent one :

$$\begin{cases} u_t = -\nabla E_{p(x)}(u), & \text{in } \Omega \times (0, T], \\ u(., t = 0) = f, & \text{in } \Omega. \end{cases} \tag{8}$$

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We consider the splitting convexity method [2] [3] to solve problem (8).

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- We split $E_{p(\cdot)}$ into a convex and a concave parts

$$E_{p(\cdot)} = E_{1,2} - E_{2,p},$$

where

$$\begin{cases} E_{1,2}(u) = \frac{c_1}{2} \int_\Omega |\Delta u|^2 \, dx + \frac{c_2}{2} \int_\Omega |u|^2 \, dx, \\ E_{2,p}(u) = -\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} \, dx - \int_\Omega W(u, f) \, dx + \frac{c_1}{2} \int_\Omega |\Delta u|^2 \, dx + \frac{c_2}{2} \int_\Omega |u|^2 \, dx, \end{cases}$$

and c_1 and c_2 are two positive constants.
For an initial data $u^0 = u(\cdot, 0)$, the CS method is obtained by :

$$\frac{u^{k+1} - u^k}{\delta t} = -\nabla_{L^2}(E_{1,2}(u^{k+1}) - E_{2,p}(u^k)).$$

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Schema

We assume :

$$\Delta\left(|\Delta u|^{p(x)-2}\Delta u\right)\approx\Delta\left(G_\epsilon(\Delta u)\right),\quad G_\epsilon(s):=(\epsilon^2+|s|^2)^{\frac{p(x)-2}{2}}s. \tag{9}$$

We obtain the following scheme

$$\frac{u^{k+1}-u^k}{\delta t}+c_1\Delta\Delta u^{k+1}+c_2u^{k+1}=-\Delta(G_\epsilon(\Delta u^k))+c_1\Delta^2u^k+c_2u^n-W'(u^k,f). \tag{10}$$

We use the Neumann boundary conditions, for all $x\in\Omega$ and $p\in]1,2]$:

$$\frac{\partial u^k}{\partial n}=\frac{\partial}{\partial n}\left(G_\epsilon(\Delta u^k)\right)=0. \tag{11}$$

We use the 2D-DCT to solve (10) :

$$\mathcal{F}(u^k_{i,j})\quad=\quad\frac{(1+\delta tc_2+\delta tc_1\theta_{i,j})\mathcal{F}(u^{k-1}_{i,j})-\mathcal{F}(W'(u^{k-1}_{i,j},f_{i,j}))-\mathcal{F}(\Delta^2_{\epsilon,p_{i,j}}u^{k-1}_{i,j})}{1+\delta tc_2+\delta tc_1\theta^2_{i,j}}, \tag{12}$$

where $\Delta^2_{\epsilon,p_{i,j}}u^{k-1}_{i,j}=\Delta((\epsilon^2+|\Delta u^{k-1}_{i,j}|^2)^{\frac{p_{i,j}-2}{2}}\Delta u^{k-1}_{i,j})$.

$f_{i,j}$, $u_{i,j}$ **and** $p_{i,j}$ **respectively act as the pixel values of the initial, restored images and the exponent function.**

Outline

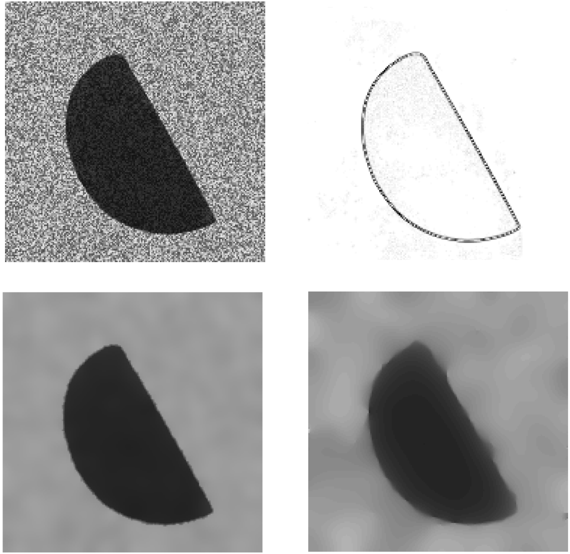
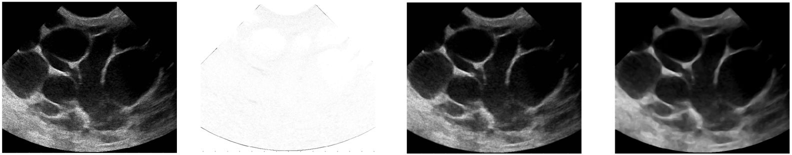


Figure : From left to right and top to bottom : Observed image, The map furnished by the exponent $p(x)$, Restored image by our model, and The TV model, respectively.



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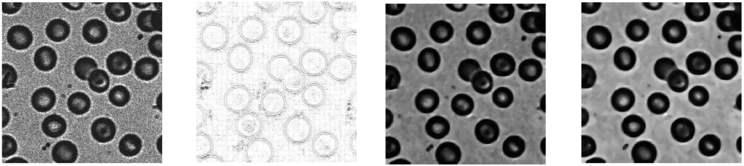

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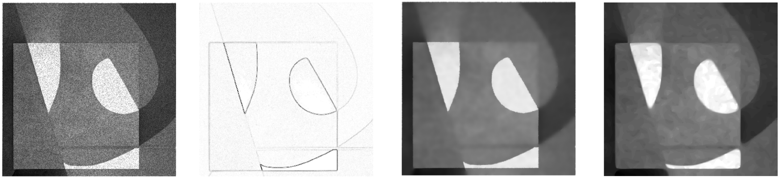

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Image	Variance	SSIM			PSNR (dB)		
		Noise	Our	TV	Noise	Our	TV
Shaped I (Fig. 2)	0.09	0.43	0.81	0.76	18.61	25.28	18.93
Mastitis (Fig. 3)	0.04	0.75	0.9	0.82	25.01	32.8	30.33
Blood cells (Fig. 4)	0.04	0.49	0.8	0.75	20.4	28.07	25
Shaped II (Fig. 5)	0.04	0.25	0.88	0.85	20.98	29.18	26

Table : Comparison between our proposed model and the TV model using the SSIM and PNSR indicators.

Outline

Conclusion

- ✓ Minimization of the nonlinear constrained $p(\cdot)$-biharmonic functionel.
- ✓ Important features detection based on the topological gradient method.
- ✓ Adaptive choice for p based on important features detected.

Prospects

- ✍ What happen if we assume that $R \neq \mathrm{id}$.
  More precisely, the Ultrasound Computer Tomography :
  $$f = Ru + \eta\sqrt{Ru},$$
  where R is the circular Radon transform.
- ✍ New numerical schema (Newton method, half quadratic method,...)

Thank you !