



Variational and numerical techniques for ultrasound medical image analysis

PhD Thesis, Applied Mathematics

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Carried out under the directions of:
Maher Moakher & Badreddine Rjaibi

July 17, 2019, Tunis, Tunisia.

Outline

- ① Ultrasound image acquisition
- ② Regularization techniques
- ③ $p(\cdot)$ -biharmonic model
- ④ $p(\cdot)$ -Kirchhoff model
- ⑤ Conclusion and prospects

Part 1: Ultrasound image acquisition

Real world image acquisition

Fields of applications



Photography



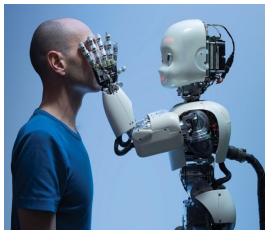
Media



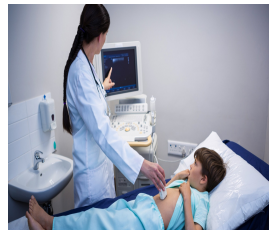
Social networks



Space

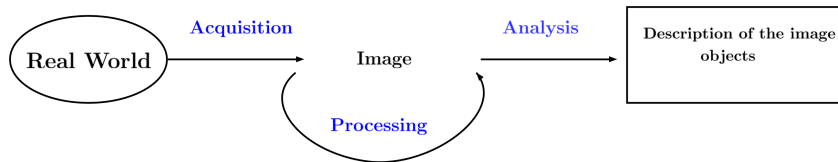


Robotic



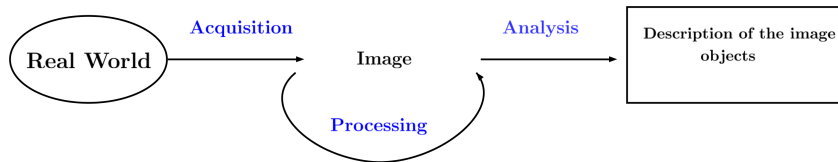
Medicine

Image processing/acquisition/analysis tasks



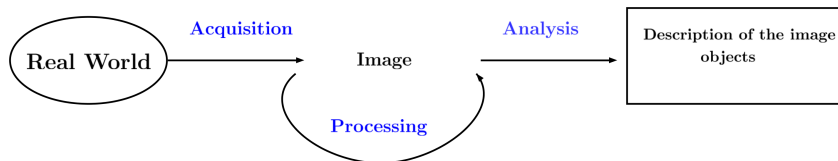
- **Acquisition:** The aim is to transform an optical image into an array of numerical data.
 - Computational photography.
 - Remote sensing.
 - **Medical image.**
 - Space image.
- **Processing:** is used to improve the quality of an image.
- **Analysis:** is the extraction of meaningful information from images.

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 - Image editing or inpainting.
 - **Image denoising.**
 - Image segmentation and classification.
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- **Analysis:** is the extraction of meaningful information from images.

- Object recognition.
- Motion detection.
- **Edges and thin-structures detection.**
- Information extraction.

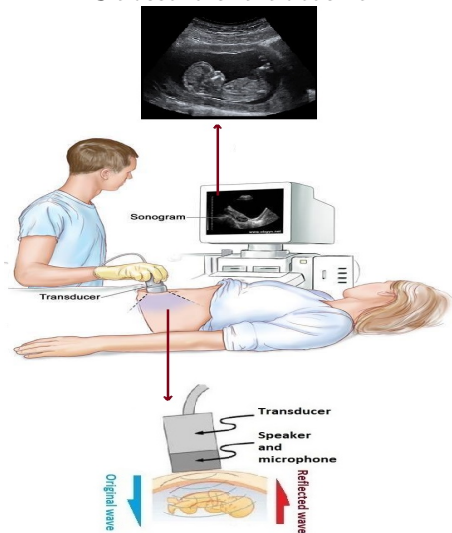
Medical ultrasound image acquisition

How ultrasound works?

Source: NIBIB.

Ultrasound image processing

Ultrasound of the abdomen



Mathematical modeling

Based on experimental measurements, the ultrasound imaging acquisition is modeled as

$$f = u + \sqrt{u}\eta, \quad (\text{M})$$

where

- f is the observed image.
- η is a **speckle noise**.
- u is the original image (**unknown**).



T. Loupas, W.-N. McDicken, and P.-L. Allan.

An adaptive weighted median filter for speckle suppression in medical ultrasonic images.

IEEE Trans. Circ. Sys., 129–135, 1989.

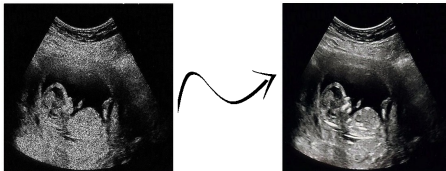
Part 2: Regularization techniques



H. Houichet, M. Moakher, and B. Rjaibi.

Noise removal and edge detection in ultrasound images by the topological gradient method.

Submitted, December 2018.



Restoration problem

Regularization procedures

In order to obtain u , we consider the following model

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How to choose the regularization term J and the parameter λ ?

- Smooth the homogeneous regions of the image.
- Unbroken the important features of the image (**edges** and **thin-structures**).

Edges: regions where the brightness changes sharply.

Thin structures: are elongated and have a width no greater than few pixels.

Total Variation model

Jin and Yang proposed the **Total Variation model**

$$\min_{u \in BV(\Omega)} \int_{\Omega} \frac{(f(x) - u(x))^2}{u(x)} dx + \lambda \int_{\Omega} |Du(x)| dx, \quad (2)$$

where

$$\int_{\Omega} |Du(x)| dx = \sup \left\{ \int_{\Omega} u(x) \operatorname{div} \phi(x) dx \mid \phi \in \mathcal{C}_c^1(\Omega), \|\phi\|_{\infty} \leq 1 \right\},$$

and $BV(\Omega)$ is the space of functions of bounded variation.



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Figure : Staircassing artifacts produced by the TV regularization term



Z. Jin and X. Yang [J. Math. Imaging Vis. 2011]

H^1 -seminorm energy

We consider the following “**Laplace model**”

$$\min_{u \in W^{1,2}(\Omega)} \frac{\lambda}{2} \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\Omega} \frac{(f(x) - u(x))^2}{u(x)} dx. \quad (3)$$

Its associate EL equation is given by

$$\begin{cases} -\lambda \Delta u + W'(u) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $W'(s)$ is the first derivative of $W(s) = \frac{(f-s)^2}{s}$.

Theorem

Let $f \in L^2(\Omega)$, with $\inf f > 0$ then, the minimization problem (3) admits a unique minimizer u in $W^{1,2}(\Omega)$ satisfying

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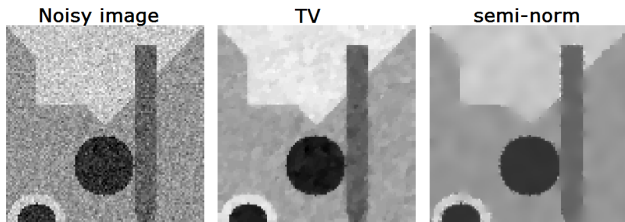
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What we gain compared to the **Total Variation model**?

- Smoothness of the image.
- Semilinear PDE.

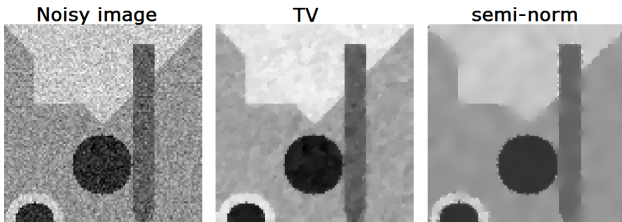
Good strategy!



The advantages of:

- Laplace model: isotropic diffusion $\Rightarrow$ smooth the flat parts.

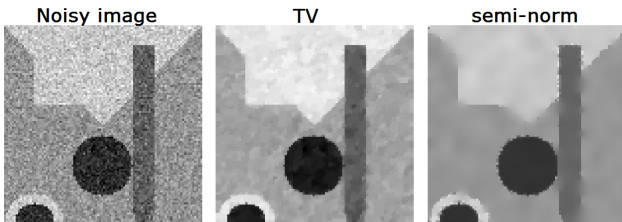
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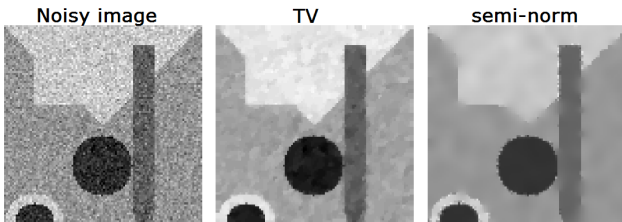


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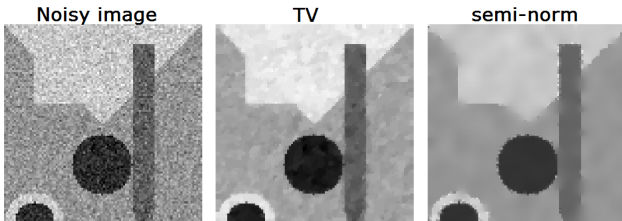
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This coefficient can be obtained by an edge detector.

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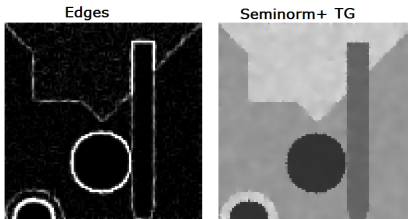


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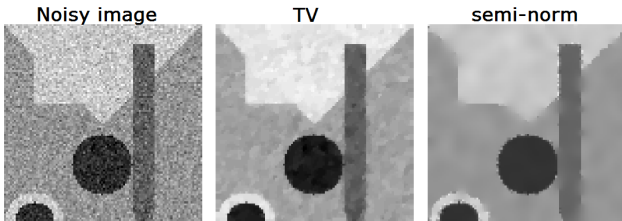
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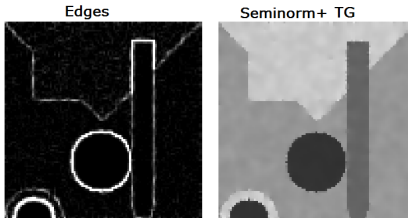
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- How the TG works?
- How to use the TG?
- Does the parameter λ depend on the TG?



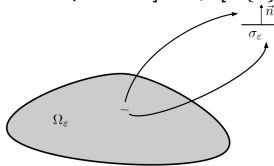
The topological gradient method

Principal

We perturb the domain Ω by:

- Inserting a small crack: Let σ be a variety of dimension 1 containing the origin. Denote by $\sigma_\varepsilon = \varepsilon\sigma$ and $\Omega_\varepsilon = \Omega \setminus \overline{\{x_0 + \sigma_\varepsilon(\vec{n})\}}$.

- Example: $\sigma =]-1, 1[\times \{0\}$



- We minimize the following cost function outside the important features:

$$J_\varepsilon(u) = \frac{\lambda}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 dx. \quad (4)$$

More generally, the variation of the cost function, has the following asymptotic expansion

$$J_\varepsilon(u) - J_0(u) = \varepsilon^2 G^S(x_0, \vec{n}) + o(\varepsilon^2),$$

$G^S(x_0, \vec{n})$ is called the topological gradient at the point x_0 .

u is the solution of:

$$\begin{cases} -\operatorname{div}(\lambda \nabla u) + W'(u) = 0, & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

The topological gradient method

Computation

We can deduce the expression of the topological gradient for a cracked domain.

$$G^S(x_0, \vec{n}) = -\lambda \pi \nabla u(x_0) \cdot \vec{n} \nabla v(x_0) \cdot \vec{n},$$

u is the solution of:

v solves the following adjoint equation:

$$\begin{cases} -\operatorname{div}(\lambda \nabla u) + W'(u) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\operatorname{div}(\lambda \nabla v) + W''(u)v = \lambda \Delta u, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

$W''(\cdot)$ is the second derivatives of $W(\cdot)$.

$G^S(x_0, \vec{n})$ corresponds to the topological gradient for second-order operator, it is sensitive to edges.



D. Auroux, L. Jaafar Belaid, and B. Rjaibi [SIAM J. Imaging Sci. 2010].



S. Amstutz [Math. Pures et Appliquées 2006].



A. Drogoul and G. Aubert [IPI 2015].



H. Houichet, M. Moakher, and B. Rjaibi [Submitted to NTMS 2018]

Results

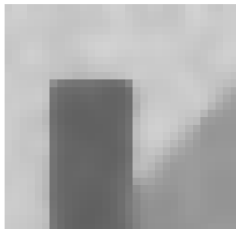
Algorithm

- ① Initialize: $\lambda = \lambda_0$.
- ② Compute u and v (solutions of direct and adjoint equations)
- ③ Compute the topological gradient $G^S(x, n)$ for each point $x_0 \in \Omega$.
- ④ Set
$$\lambda = \begin{cases} \lambda_1 & \text{if } x_0 \in \{x \in \Omega \text{ such that } |G^S(x, n)| < \alpha\}, \\ \lambda_0 & \text{otherwise } (\lambda_1 \ll \lambda_0). \end{cases}$$
- ⑤ Compute u with the new value of λ . 

Results

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The Laplace model

PSNR= 31.22dB,
SSIM=0.88.



The TV model

PSNR= 32.53dB,
SSIM=0.91.



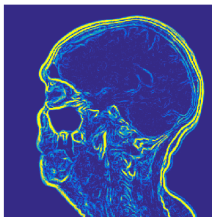
Our approach

PSNR= 35.37dB,
SSIM=0.95.

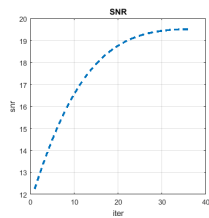
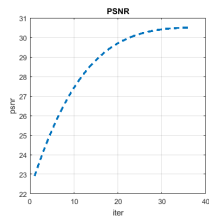
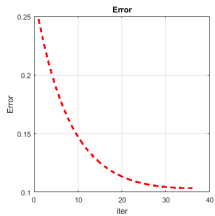
Noisy image



Edges



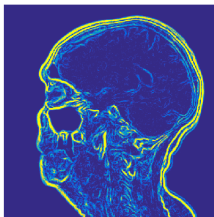
Restored image



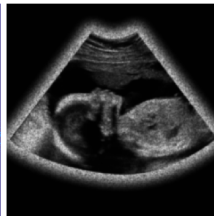
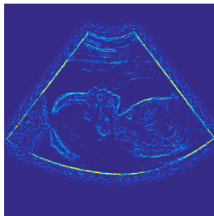
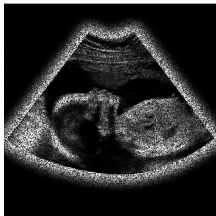
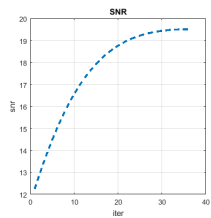
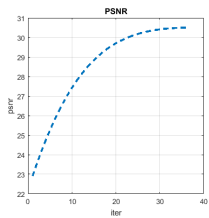
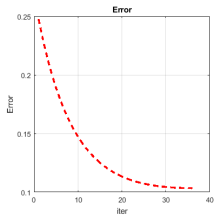
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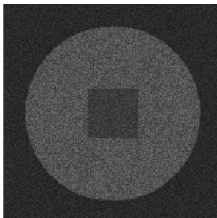
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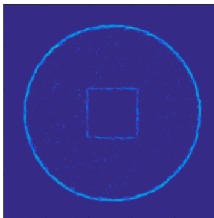
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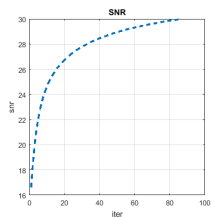
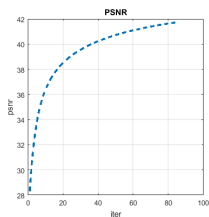
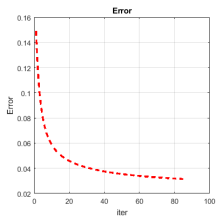
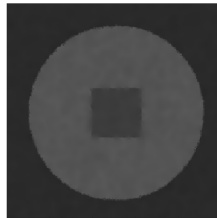
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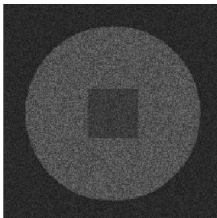
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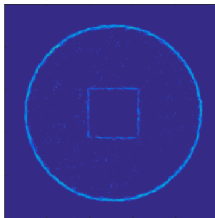
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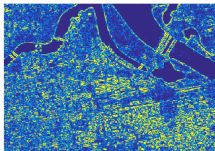
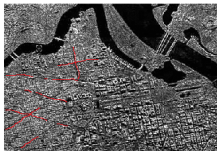
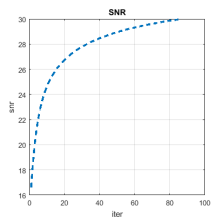
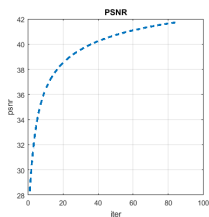
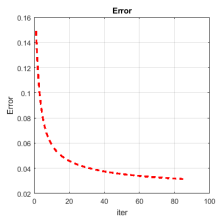
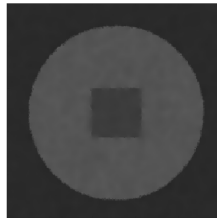
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Edges



Restored image



What is the difference between thin structures and contours?

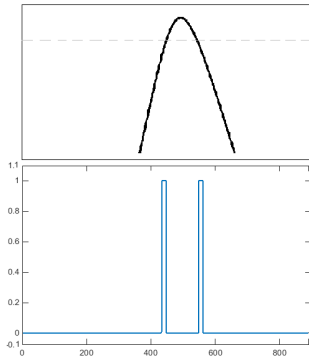
What is the difference between thin structures and contours?

Thin structures

Discontinuities without jump of intensity.

The size depends on the resolution of the image.

Profile in the form of the Dirac function.

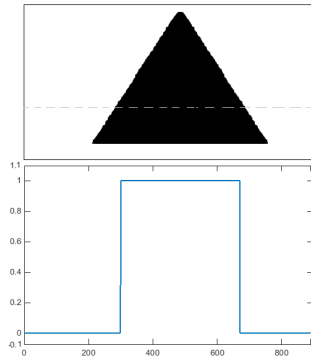


Edges

Intensity jump through the discontinuity.

It separates two regions (or more) of the image.

Profile in the form of the Heaviside function.



Best operator to detect thin structures

Example (G. Aubert & A. Drogoul, COCV 2015)

- **(point)** Let $a \in \mathbb{R}^2$, $g_n(x) = \frac{1}{\pi n} e^{-\frac{\|x-a\|^2}{n}}$, then, $g_n \rightarrow \delta_a$, in $\mathcal{D}'(\mathbb{R}^2)$, as $n \rightarrow 0$. Moreover,

$$\nabla g_n(a) = [0, 0]^T, \quad \text{and} \quad \nabla^2 g_n(a) = -\frac{2}{\pi n^2} \mathbf{I}.$$

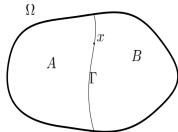
- **(filament)** Let Γ be a smooth closed curve of $\mathbb{R}^2$ delimiting two sub-domains A and B of Ω .

Let $g_n(x) = \frac{1}{\sqrt{\pi n}} e^{-\frac{\Phi(x)^2}{n}}$ where Φ be the signed distance to Γ defined by:

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Then $g_n \rightarrow \delta_\Gamma$ in $\mathcal{D}'(\mathbb{R}^2)$ as $n \rightarrow 0$. Moreover, for all $x \in \Gamma$,

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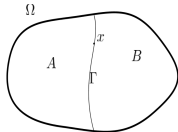
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The second order derivative is necessary to detect these singular values (filaments and points).

Best operator to detect thin structures

Example (G. Aubert & A. Drogoul, COCV 2015)

- **(point)** Let $a \in \mathbb{R}^2$, $g_n(x) = \frac{1}{\pi n} e^{-\frac{\|x-a\|^2}{n}}$, then, $g_n \rightarrow \delta_a$, in $\mathcal{D}'(\mathbb{R}^2)$, as $n \rightarrow 0$. Moreover,

$$\nabla g_n(a) = [0, 0]^T, \quad \text{and} \quad \nabla^2 g_n(a) = -\frac{2}{\pi n^2} \mathbf{I}.$$

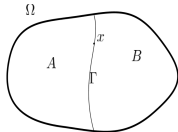
- **(filament)** Let Γ be a smooth closed curve of $\mathbb{R}^2$ delimiting two sub-domains A and B of Ω .

Let $g_n(x) = \frac{1}{\sqrt{\pi n}} e^{-\frac{\Phi(x)^2}{n}}$ where Φ be the signed distance to Γ defined by:

$$\Phi(x) = \text{dist}(x, B) - \text{dist}(x, A).$$

Then $g_n \rightarrow \delta_\Gamma$ in $\mathcal{D}'(\mathbb{R}^2)$ as $n \rightarrow 0$. Moreover, for all $x \in \Gamma$,

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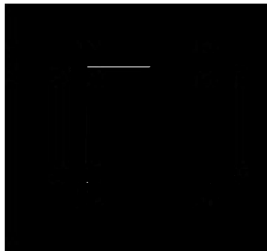


The second order derivative is necessary to detect these singular values (filaments and points).

- For the edges, we use $J(u) = \int_{\Omega} |\nabla u|^2 dx$.
- For thin structure a 4th order operator is well recommended, e.g., $J(u) = \int_{\Omega} |\Delta u|^2 dx$.

Best operator to detect thin structures

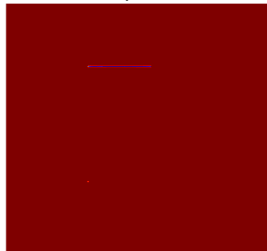
Point and filament



Objects detection using
2nd order operator

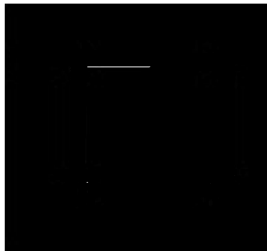


Objects detection using
4nd order operator



Best operator to detect thin structures

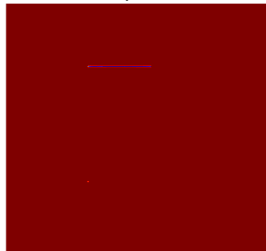
Point and filament



Objects detection using
2nd order operator



Objects detection using
4th order operator



A fourth order operator is a well suitable operator to detect thin structures.

Part 3: $p(\cdot)$ -biharmonic model



H. Houichet, and A. Theljani.

Well-posedness of high-order non-standard parabolic partial differential equation for image denoising.

In revision at App. Math., April 2019.

$p(\cdot)$ -biharmonic model

We consider the following constrained high-order minimization problem

$$\min_{u \in X(\Omega)} \underbrace{\int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx}_{\text{Regularizer}} + \lambda \underbrace{\int_{\Omega} \frac{(f-u)^2}{u} dx}_{\text{Fidelity}} \quad (5)$$

$$\text{Subject to } 0 < \inf_{x \in \overline{\Omega}} f \leq u \leq \sup_{x \in \overline{\Omega}} f, \quad (6)$$

where

- $p : \Omega \rightarrow]1, 2]$ is continuous function so-called **exponent function** and satisfies

$$1 < p^- := \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \sup_{x \in \Omega} p(x) \leq 2.$$

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- $X(\Omega)$ is a **variable exponent space** defined as

$$X(\Omega) = \left\{ u \in W^{2,p(x)} : \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \right\}$$

which is endowed with the norm $\|u\|_X := \|u\|_{p(x)} + \|\Delta u\|_{p(x)}$.

- $\|u\|_{p(x)} = \inf \left\{ \tau > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}$ is a norm in

$$L^{p(x)} = \{ u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \}.$$

$(X(\Omega), \|\cdot\|_X)$ is a closed subspace of the separable and reflexive Banach space $(W^{2,p(x)}(\Omega), \|\cdot\|_X)$.

Unconstrained problem

We transform the constrained minimization problem into

$$\min_{u \in X(\Omega)} \{E_{p(x)}(u) := J(u) + W(u, f)\}, \quad (7)$$

where

$$J(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx,$$
$$W(u, f) = \lambda \int_{\Omega} \frac{(f - u)^2}{u} dx + \underbrace{\frac{c}{2} \|\min(0, u - m)\|_{L^2}^2 + \frac{c}{2} \|\max(0, u - M)\|_{L^2}^2}_{\text{Moreau-Yosida penalization}}.$$

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🌀 Well-posedness of the minimization problem

Theorem

Let $f \in X(\Omega) \cap L^2(\Omega)$. Then, the minimization problem (7) admits a unique minimizer u in $X(\Omega) \cap L^2(\Omega)$.

Sketch of proof

Existence:

- We consider a minimizing sequence $(u_n)_n \subset X(\Omega)$. Then, there exists $C > 0$ such that

$$E_{p(x)}(u_n) \leq C.$$

- ▶ We prove $\|\Delta u_n\|_{L^{p(\cdot)}}$ is uniformly bounded (clear because $J(u_n) \leq C$).
- ▶ We prove that $\|u_n\|_{L^{p(\cdot)}}$ is uniformly bounded using

$$\begin{aligned}\int_{\Omega} \max(0, u_n - M)^2 \, dx &= \int_{\{u_n > M\}} (u_n - M)^2 \, dx \leq C, \\ \int_{\Omega} \min(0, u_n - m)^2 \, dx &= \int_{\{u_n < m\}} (u_n - m)^2 \, dx \leq C,\end{aligned}$$

and the continuous embedding: $1 < p(x) \leq 2 \Rightarrow L^2 \hookrightarrow L^{p(x)}$.

- u_n is bounded in $X(\Omega)$. Then $\exists u \in X(\Omega) : u_n \xrightarrow[n \rightarrow \infty]{} u$ weakly in $X(\Omega)$.
- Apply the lower semi-continuous of $E_{p(x)}$ in $X(\Omega)$ and Fatou's lemma.

Uniqueness:

- The strictly convex of $E_{p(x)}$.

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Uniqueness:

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We transform (7) into an evolution problem...

$$\begin{cases} u_t + \Delta (|\Delta u|^{p(x)-2} \Delta u) + W'(u, f) = 0, & \text{in } \Omega \times (0, T], \\ \frac{\partial}{\partial n} (|\Delta u|^{p(x)-2} \Delta u) = \frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega. \\ u(\cdot, t=0) = f, & \text{in } \Omega. \end{cases} \quad (8)$$

Well-posedness of the evolution problem

Theorem

The initial-boundary value problem (8) admits a unique weak solution u , which satisfies:

(i) $u \in L^\infty(0, T; L^2(\Omega)) \cap L^{p^-}(0, T; X(\Omega))$, $\Delta u \in L^{p(x)}(Q_T)$,
 $\partial_t u \in L^{p^-}(0, T; X(\Omega)^*)$, $u(0) = f$.

(ii) For any $\phi \in L^\infty(0, T; L^2(\Omega)) \cap L^{p^-}(0, T; X(\Omega))$, we have

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle dt + \int_0^T \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \phi \, dx dt + \int_0^T \int_\Omega W'(u, f) \phi \, dx dt = 0.$$

(iii) $\|u\|_{L^\infty(0, T; L^2(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^{p^-}(0, T; X(\Omega)')} + \|u\|_{L^{p^-}(0, T; X(\Omega))} \leq C.$



A. Theljani, Z. Belhachmi, and M. Moakher [Non. Ana. 2019].



H. Houichet, A. Theljani [In revision at App. Math. 2019]

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Proposition $u(x, t)$ satisfies:

- **Extremum principle** $m \leq u(x, t) \leq M$, for a.e $(x, t) \in Q_T$.
- **Mean value**

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = \frac{1}{|\Omega|} \int_\Omega f \, dx.$$



A. Theljani, Z. Belhachmi, and M. Moakher [Non. Ana. 2019].



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Finite Difference method (FDM)

In order to solve the evolutionary equation, we use the following semi-implicit scheme:

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{h} = -W'(u_{i,j}^k, f_{i,j}) - \Delta(G_\epsilon(\Delta u_{i,j}^k)), \quad (9)$$

where

$$G_\epsilon(s) = \frac{s}{\sqrt{\epsilon^2 + |s|^{2-p_{i,j}}}}$$

with $u_{i,j}^k(x) = u(x_i, y_j, t^k)$, $f_{i,j}(x) = f(x_i, y_j)$, $p_{i,j}(x) = p(x_i, y_j)$, $t^k = kh$ and $(i, j) \in \{1, \dots, r\} \times \{1, \dots, c\}$.

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We use the following indicator error: **The Laplacian of Gaussian (LoG)**

$$\text{LoG}^k(u^k) = \Delta(\mathcal{G}_\sigma * u^k) = \Delta \mathcal{G}_\sigma * u^k, \quad (10)$$

where $\mathcal{G}_\sigma(\mathbf{x})$ is the Gaussian kernel.

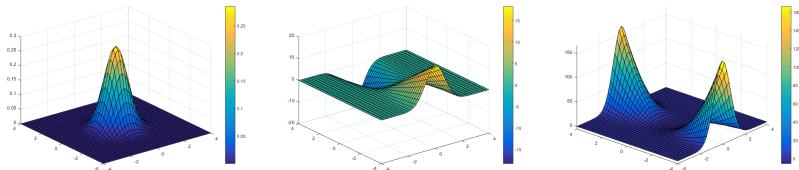


Figure : The Gaussian function, its first derivative and its second derivative.

Image



Sobel



LoG



Sobel operator

- It is based on extrema of the first order derivatives.
- The objects are become fragmented.
- It uses tow kernels.

LoG

- ZC are based on second order derivative.
- It forms connected and closed objects.
- The objects are become very tiny.
- It uses only one kernel.

Both of them work with convolution operator.

$$G_x = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} * u, G_y = G_x^T * u.$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} * u.$$

Algorithm 2: Adaptive strategy for the choice of p

- ① Given $u^0 = f$.
- ② Compute LoG_σ^0 and detection zero-crossing for the image u^0 .
- ③ Threshold the zero-crossings to keep only the strong ones for the image u^0 (denoting by $\mathcal{ZC}_s^0(x)$) i.e.,

$$\mathcal{ZC}_s := \{|\text{LoG}(u(x))| \geq \delta, \text{ for all } x \in \Omega, \delta > 0\},$$

where δ is small threshold parameter.

We use the following formula for the exponent $p(\cdot)$:

$$p(x) = 1 + e^{-\mu |\mathcal{ZC}_s(x)|}, \text{ for all } x \in \Omega, \text{ and } \mu > 0.$$

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- In homogeneous regions: $\mathcal{ZC}_s(x) \approx 0 \Rightarrow p(x)$ closes to 2 (**Strong diffusion**).
- Near important features: $\mathcal{ZC}_s(x)$ is very important $\Rightarrow p(x)$ closes to 1 (**weak diffusion**).

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Algorithm 3: Main algorithm for image restoration

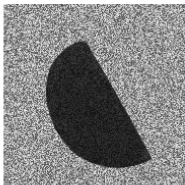
- ① **Initialization** $k = 1$, given u^0 and λ .
- ② **Compute** $p(x) = 1 + e^{-\mu|\mathcal{ZC}_s^{k-1}(x)|}$ by Algorithm 2.
- ③ **Solve** (9) by computing u^k .
- ④ **Repeat** steps 2 and 3 until the stopping criterion is satisfied

$$\text{ReErr} = \frac{\|u^k - R\|}{\|R\|} \leq \text{Tot}.$$

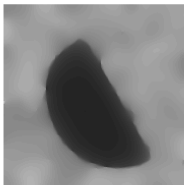
We compare with

- **TV:** The Total Variation model.
- **TL:** The Total-Laplace model i.e., for $p(x) \equiv 1$, $\forall x \in \Omega$.
- **BL:** The bi-Laplace model i.e., for $p(x) \equiv 2$, $\forall x \in \Omega$.

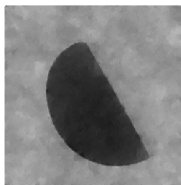
Numerics



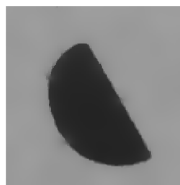
Noisy image
PSNR= 13.61dB,
SSIM=0.16.



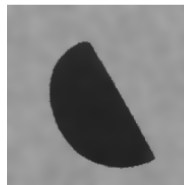
The TV model
PSNR= 18.93dB,
SSIM=0.76.



The TL model
PSNR= 18.03dB,
SSIM=0.75.

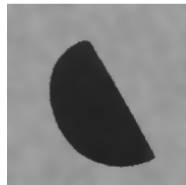
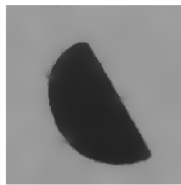
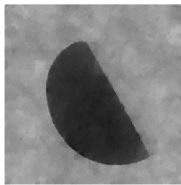
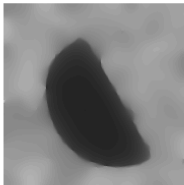
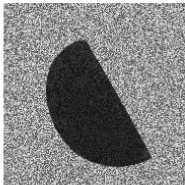


The BL model
PSNR= 24.16dB,
SSIM=0.78.



Our approach
PSNR= 25.28dB,
SSIM=0.81.

Numerics



Noisy image

PSNR= 13.61dB,
SSIM=0.16.

The TV model

PSNR= 18.93dB,
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The TL model

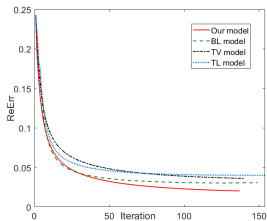
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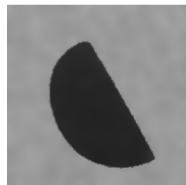
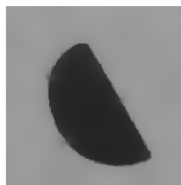
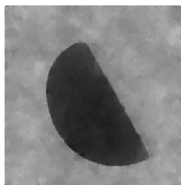
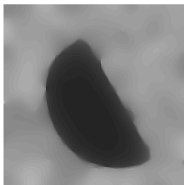
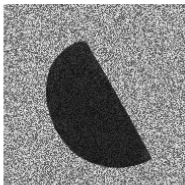
Our approach

PSNR= 25.28dB,
SSIM=0.81.



The relative error

Numerics



Noisy image

PSNR= 13.61dB,
SSIM=0.16.

The TV model

PSNR= 18.93dB,
SSIM=0.76.

The TL model

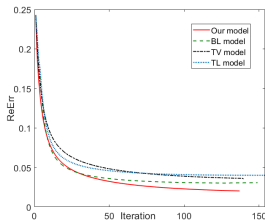
PSNR= 18.03dB,
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The BL model

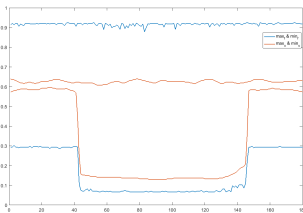
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SSIM=0.78.

Our approach

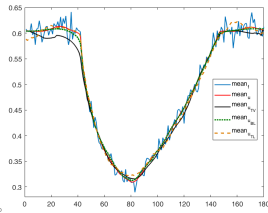
PSNR= 25.28dB,
SSIM=0.81.



The relative error

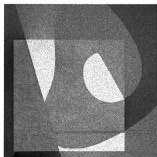


The extremum Principle
 $\inf f \leq u \leq \sup f$



The mean value
 $\frac{1}{|\Omega|} \int_{\Omega} u dx = \frac{1}{|\Omega|} \int_{\Omega} f dx$

Noisy image



SSIM=0.75.

The TV model



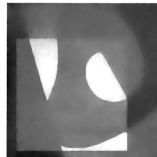
SSIM=0.82.

The TL model



SSIM=0.86.

The BL model

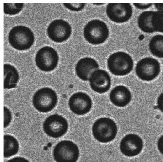


SSIM=0.8.

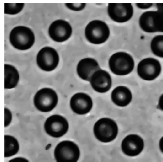
Our approach



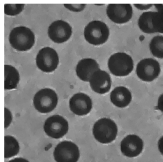
SSIM=0.9.



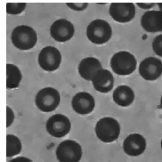
SSIM=0.25.



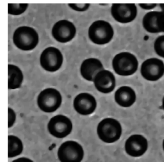
SSIM=0.85.



SSIM=0.81.



SSIM=0.84.



SSIM=0.88.



SSIM=0.69.



SSIM=0.85.



SSIM=0.84.



SSIM=0.86.



SSIM=0.87.

Part 4: $\rho(\cdot)$ -Kirchhoff model



H. Houichet, A. Theljani, B. Rjaibi, and M. Moakher.

A nonstandard higher-order variational model for speckle noise removal and thin-structure detection.

Accepted on JMS, April 2019.

Is it worth to combine the 2nd and 4th order models?

$p(\cdot)$ -Kirchhoff energy

We consider the $p(\cdot)$ -Kirchhoff energy

$$\min_{u \in X(\Omega)} \{E_{p(x)}(u) := J^{p,q}(u) + W(u, f)\}, \quad (11)$$

where

$$J^{p,q}(u) = \alpha \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \beta \int_{\Omega} \frac{1}{q(x)} |\nabla u|^{q(x)} dx,$$
$$W(u, f) = \lambda \int_{\Omega} \frac{(f - u)^2}{u} dx + \iota_C(u).$$

where ι_C is the indicator function of $C = [m, M]$ and which is given by

$$\iota_C(s) = \begin{cases} 0, & \text{if } s \in C, \\ +\infty, & \text{if } s \notin C. \end{cases}$$

The exponents p and q satisfy $1 < p, q \leq 2$.

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Theorem

Let $f \in X(\Omega) \cap L^2(\Omega)$. Then, the minimization problem (11) admits a unique minimizer u in $X(\Omega) \cap W^{1,q(x)}(\Omega)$.

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How to choose the exponents $p(x)$ and $q(x)$?

Topological gradient

- We minimize the following cost function (with $p(\cdot) \equiv q(\cdot) \equiv 2$) outside the important features:

$$J_\varepsilon(u) = \frac{\alpha}{2} \int_{\Omega_\varepsilon} |\Delta u|^2 \, dx + \frac{\beta}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 \, dx, \quad (12)$$

- We compute the following asymptotic expansion

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^2 G(x_0, \vec{n}) + o(\varepsilon^2),$$

- We obtain:

- For $\beta = 0$, the topological gradient for a **fourth-order PDE**

$$G^F(x_0, \vec{n}) = -\frac{2\pi}{3} \nabla(\nabla u(x_0) \cdot \vec{n}) \cdot \vec{n} \nabla(\nabla v(x_0) \cdot \vec{n}) \cdot \vec{n},$$

where u and v are the solutions of the direct and adjoint equations, respectively.

- For $\alpha = 0$, the topological gradient for a **second-order PDE**

$$G^S(x_0, \vec{n}) = -\beta \pi \nabla u(x_0) \cdot \vec{n} \nabla v(x_0) \cdot \vec{n}.$$

- 1) $G^S(x_0, \vec{n})$ is sensitive to edges.
- 2) $G^F(x_0, \vec{n})$ is sensitive to thin structures and points.



G. Aubert and A. Drogoul [COCV 2015].



Houichet, Theljani, Rjaibi, and Moakher [Accepted in JMS, April 2019].

The choice of p and q is based on the following algorithm...

Algorithm 4: Main algorithm for image restoration

- ① Given f and λ .
- ② For $p(\cdot) \equiv q(\cdot) \equiv 2$, compute the topological gradient indicators $G^F(x_0, n)$ and $G^S(x_0, n)$ for each point $x_0 \in \Omega$.
- ③ Update $p(\cdot)$ and $q(\cdot)$ to obtain new exponents $p_a(\cdot)$ and $q_a(\cdot)$ i.e.,

$$p_a(x) = 1 + \exp(-\kappa |G^F(x, \vec{n})|), \quad q_a(x) = 1 + \exp(-\kappa |G^S(x, \vec{n})|), \quad \forall x \in \Omega,$$

where $\kappa > 0$ is a constant.

- ④ Compute u solution of the initial minimization problem (11) for $p(\cdot) \equiv p_a(\cdot)$ and $q(\cdot) \equiv q_a(\cdot)$.

-
- In homogeneous regions: $G^F(x, \vec{n}) \approx 0$ and $G^S(x, \vec{n}) \approx 0 \Rightarrow p_a(x)$ and $q_a(x)$ close to 2 (**Strong diffusion**).
 - Near important features: $G^F(x, \vec{n})$ and $G^S(x, \vec{n})$ are very important $\Rightarrow p_a(x)$ and $q_a(x)$ close to 1 (**weak diffusion**).

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How to numerically solve the minimization problem (11)?

Split Bregman method

Formulation

$$\min_{u \in \mathbb{R}^r \times c} \sum_{i=1}^r \sum_{j=1}^c \left(\frac{\alpha}{p(i,j)} |\Delta u(i,j)|^{p(i,j)} + \frac{\beta}{q(i,j)} |\nabla u(i,j)|^{q(i,j)} + \lambda \frac{(u(i,j) - f(i,j))^2}{u(i,j)} + \iota_C(u(i,j)) \right), \quad (11)$$

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- We need to split the L^p , L^q and the data-fitting term of (11).

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- We need to split the L^p , L^q and the data-fitting term of (11).
- We introduce the new variables

$$\mathbf{v} = u, \quad \mathbf{m} = \Delta u, \quad \text{and} \quad \mathbf{n} = \nabla u.$$

- We wish to solve the constrained problem

$$\min_{u \in \mathbb{R}^{r \times c}} F(u) + G(X), \quad \text{such that } \mathbf{A}u = X, \quad (13)$$

where $X = [\mathbf{v}, \mathbf{m}, \mathbf{n}]^T$, $\mathbf{A} = [\mathbf{I}, \Delta, \nabla]^T$ and $G(X) = G_1(\mathbf{v}) + G_2(\mathbf{m}) + G_3(\mathbf{n})$ with

$$\begin{aligned} F(u) &= \iota_C(u) & G_1(\mathbf{v}) &= \lambda \sum_{i=1}^r \sum_{j=1}^c \frac{(\mathbf{v}(i,j) - f(i,j))^2}{\mathbf{v}(i,j)} \\ G_2(\mathbf{m}) &= \alpha \|\mathbf{m}\|_p & G_3(\mathbf{n}) &= \beta \|\mathbf{n}\|_q \end{aligned}$$



T. Goldstein and S. Osher [SIAM J. Imag. Sci. 2008].

Split Bregman method

Scheme

$$\min_{u \in \mathbb{R}^r \times c} F(u) + G(X), \text{ such that } \mathbf{A}u = X,$$

- We add an L^2 penalty term to get an unconstrained problem

$$(u_{k+1}, X_{k+1}) = \operatorname{argmin}_{u, X} F(u) + G(X) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2. \quad (14)$$

$$b_{k+1}^0 = b_k^0 + u_{k+1} - v_{k+1},$$

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- We can break (14) down into four easy subproblems:

$$\begin{aligned} \text{Subproblem 1 : } u_{k+1} = \operatorname{argmin}_u & F(u) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 \\ & + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2, \end{aligned}$$

$$\text{Subproblem 2 : } v_{k+1} = \operatorname{argmin}_v G_1(v) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2,$$

$$\text{Subproblem 3 : } \mathbf{m}_{k+1} = \operatorname{argmin}_{\mathbf{m}} G_2(\mathbf{m}) + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2,$$

$$\text{Subproblem 4 : } \mathbf{n}_{k+1} = \operatorname{argmin}_{\mathbf{n}} G_3(\mathbf{n}) + \frac{\lambda_2}{2} \|b_k^2 + \Delta u - \mathbf{n}\|_2^2,$$

Subproblem 2 : $v_{k+1} = \underset{v}{\operatorname{argmin}} G_1(v) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2$

whose solution fulfills

$$Av^3 + Bv^2 + C = 0,$$

where $A = \lambda_0$, $B = \lambda + \lambda_0(b_k^0 + u_{k+1})$ and $C = -\lambda f^2$.

Then, v_{k+1} is the real root of this polynomial.

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Subproblem 4 :
$$\mathbf{n}_{k+1} = \underset{\mathbf{n}}{\operatorname{argmin}} G_3(\mathbf{n}) + \frac{\lambda_0}{2} \|b_k^2 + \Delta u - \mathbf{n}\|_2^2,$$

Subproblems 3 and 4 can be solved efficiently using the shrinkage operator

$$\mathbf{m}_{k+1} = \max \left\{ |b_k^1 + \Delta u_{k+1}| - \frac{|b_k^1 + \Delta u_{k+1}|^{p(\cdot)-1}}{\lambda_1 p(\cdot)}, 0 \right\} \frac{b_k^1 + \Delta u_{k+1}}{|b_k^1 + \Delta u_{k+1}|},$$

and

$$\mathbf{n}_{k+1} = \max \left\{ |b_k^2 + \nabla u_{k+1}| - \frac{|b_k^2 + \nabla u_{k+1}|^{q(\cdot)-1}}{\lambda_2 q(\cdot)}, 0 \right\} \frac{b_k^2 + \nabla u_{k+1}}{|b_k^2 + \nabla u_{k+1}|},$$

respectively.

Subproblem 1 :
$$u_{k+1} = \underset{u}{\operatorname{argmin}} F(u) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2$$

To solve this problem, we use

$$\text{Subproblem 1 : } u_{k+1} = \underset{u}{\operatorname{argmin}} F(u) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 \\ + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2$$

To solve this problem, we use an inner split-Bregman scheme

Subproblem 1 :
$$u_{k+1} = \underset{u}{\operatorname{argmin}} F(u) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2$$

To solve this problem, we use an inner split-Bregman scheme:

- We introduce a new constrained

$$w = u.$$

- We add an L^2 penalty term to get an unconstrained problem

$$(u_{k+1}, w_{k+1}) = \underset{u, w}{\operatorname{argmin}} F(w) + \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2 + \frac{\lambda_3}{2} \|b_k^3 + u - w_k\|_2^2.$$

$$b_{k+1}^3 = b_k^3 + u_{k+1} - w_{k+1}.$$

- We can break (14) down into two easy subproblems:

u -subproblem

$$u_{k+1} = \underset{u}{\operatorname{argmin}} \frac{\lambda_0}{2} \|b_k^0 + u - v\|_2^2 + \frac{\lambda_1}{2} \|b_k^1 + \Delta u - \mathbf{m}\|_2^2 + \frac{\lambda_2}{2} \|b_k^2 + \nabla u - \mathbf{n}\|_2^2 + \frac{\lambda_3}{2} \|b_k^3 + u - w_k\|_2^2.$$

whose solution fulfills

$$u_{k+1} = \frac{\lambda_0(b_k^0 - v_k) + \lambda_1 \Delta^2(\mathbf{m}_k - b_k^1) + \lambda_2 \operatorname{div}(b_k^2 - \mathbf{n}_k) + \lambda_3(b_k^3 - w_k)}{\lambda_1 \Delta^2 - \lambda_2 \Delta + (\lambda_0 + \lambda_3)}.$$

w-subproblem

$$w_{k+1} = \underset{w}{\operatorname{argmin}} F(w) + \frac{\lambda_3}{2} \|b_k^3 + u - w_k\|_2^2.$$

its solution can be obtained by the proximal operator

$$w_{k+1} = \operatorname{prox}(b_k^3 + u_{k+1}) = \begin{cases} b_k^3 + u_{k+1}, & \text{if } b_k^3 + u_{k+1} \in [m, M], \\ M, & \text{if } b_k^3 + u_{k+1} > M, \\ m, & \text{if } b_k^3 + u_{k+1} < m. \end{cases}$$

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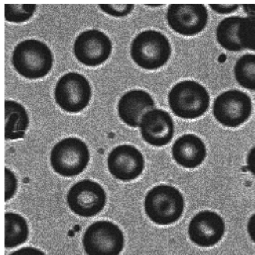
$$w_{k+1} = \operatorname{prox}(b_k^3 + u_{k+1}) = \begin{cases} b_k^3 + u_{k+1}, & \text{if } b_k^3 + u_{k+1} \in [m, M], \\ M, & \text{if } b_k^3 + u_{k+1} > M, \\ m, & \text{if } b_k^3 + u_{k+1} < m. \end{cases}$$

Proposition: (Convergence)

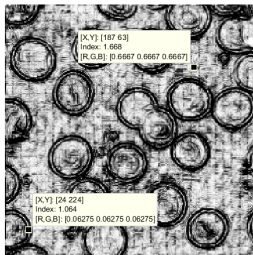
For any $\lambda_i > 0$ and all starting values, the sequence (X_k) , (w_k) and (b_k^i) converge. Furthermore, the sequence (u_k) calculated by the u -subproblem converges to the solution of the problem (11).

Numerical results

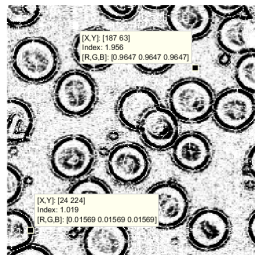
Blood cells



Noisy image



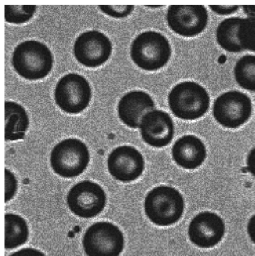
The exponent $p(x)$



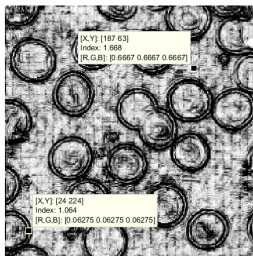
The exponent $q(x)$

Numerical results

Blood cells



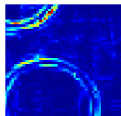
Noisy image



The exponent $p(x)$

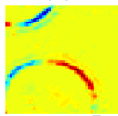
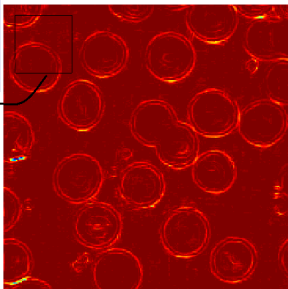


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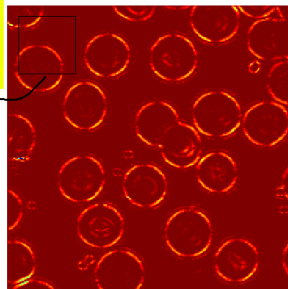
Zoom

TG for fourth-order PDE



Zoom

TG for second-order PDE



Comparison

We compare our model with the following classical ones:

- The **TVL model** i.e., $p(\cdot) \equiv q(\cdot) \equiv 1$:

$$\min_{u,v,\mathbf{m},\mathbf{n}} \alpha \|\mathbf{m}\|_1 + \beta \|\mathbf{n}\|_1 + \lambda \sum_{i=1}^r \sum_{j=1}^c \frac{(v(i,j) - f(i,j))^2}{v(i,j)} + \iota_C(u).$$

such that $u = v$, $\Delta u = \mathbf{m}$, $\nabla u = \mathbf{n}$.

- The **biharmonic model** i.e., $p(\cdot) \equiv q(\cdot) \equiv 2$:

$$\min_{u,v,\mathbf{m},\mathbf{n}} \alpha \|\mathbf{m}\|_2 + \beta \|\mathbf{n}\|_2 + \lambda \sum_{i=1}^r \sum_{j=1}^c \frac{(v(i,j) - f(i,j))^2}{v(i,j)} + \iota_C(u),$$

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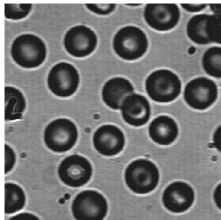
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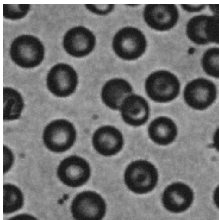
- The **biharmonic model** i.e., $p(\cdot) \equiv q(\cdot) \equiv 2$:

$$\min_{u,v,\mathbf{m},\mathbf{n}} \alpha \|\mathbf{m}\|_2 + \beta \|\mathbf{n}\|_2 + \lambda \sum_{i=1}^r \sum_{j=1}^c \frac{(v(i,j) - f(i,j))^2}{v(i,j)} + \iota_C(u),$$

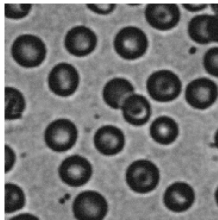
such that $u = v$, $\Delta u = \mathbf{m}$, $\nabla u = \mathbf{n}$.



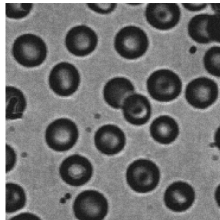
Original image



SSIM=0.73, SNR=17.33dB
TVL model



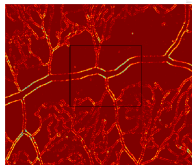
SSIM=0.76, SNR=17.67dB
Bi-harmonic model



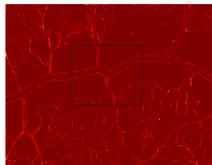
SSIM=0.81, SNR=18.93dB
p(.)-Kirchhoff model

Numerical examples

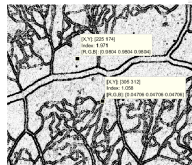
Blood vessel detection



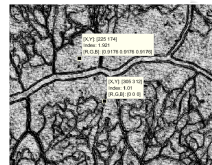
TG for 2nd order PDE



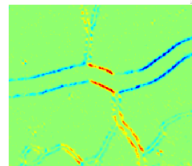
TG for 4th order PDE



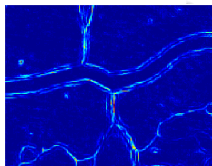
The exponent q



The exponent p



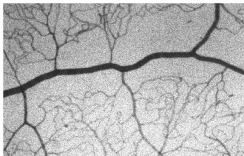
Zoom of 2nd order TG



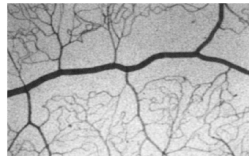
Zoom of 4th order TG

Numerical examples

Blood vessel detection



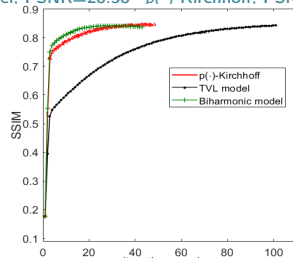
Noisy image, PSNR=21.23



TV model, PSNR=27.79

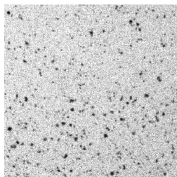


Biharmonic model, PSNR=28.58 $p(\cdot)$ -Kirchhoff, PSNR=29.3

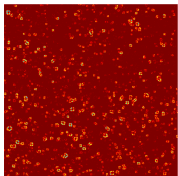


Numerical examples

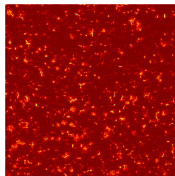
Points detection in biological image



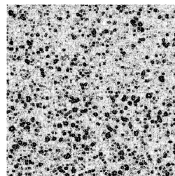
Noisy image
(ssim=0.18,
psnr=16.77dB).



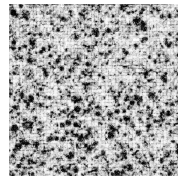
TG for 2nd order
PDE



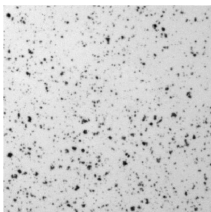
TG for 4th order
PDE



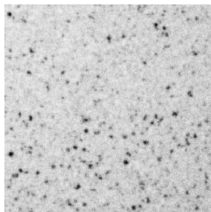
The exponent q



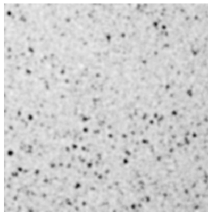
The exponent p



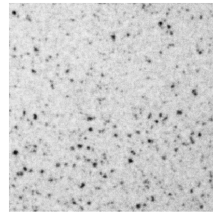
Original image



TVL model
ssim=0.67, psnr=24.41dB.



Bihrmonic model
(ssim=0.66, psnr=23.44dB).



$p(\cdot)$ -Kirchhoff model
(ssim=0.7, psnr=25.74dB).

Numerical examples

Ultrasound image



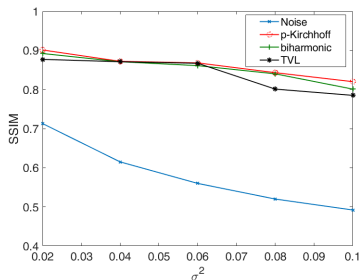
TVL model (ssim=0.867,
psnr=29.91dB).



Biharmonic model
(ssim=0.861, psnr=29.36dB).

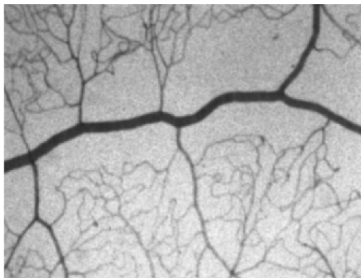


$p(\cdot)$ -Kirchhoff model
(ssim=0.868, psnr=30dB).



The sensitivity of noise level as a function of the SSIM for the three models.

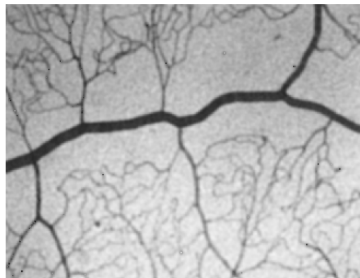
Comparison between the proposed models



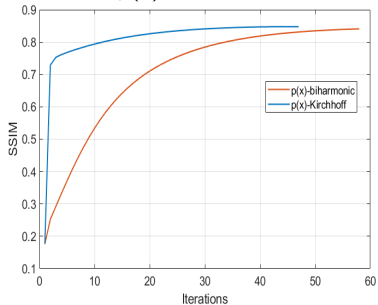
Model 1: $p(x)$ -biharmonic model

	Noisy	Model 1	Model 2
PSNR	17.39	28.97	29.3
SNR	14.64	26.06	26.52
SSIM	0.17	0.840	0.847
Iteration	-	58	48
CPU (s)	-	174	251

Table : Comparison between $p(x)$ -Kirchhoff and $p(x)$ -biharmonic models



Model 2: $p(x)$ -Kirchhoff model



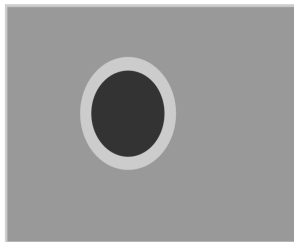
Conclusion and prospects

- ✓ We proposed novel high order variational models for image restoration.
 - ✓ The smoothness of the solution is driven from the variable exponent functions which are chosen based on important features location.
 - ✓ Edges and thin-structures are detected based on second and fourth-order operators (topological gradient and LoG).
 - ✓ Fast algorithms are developed for the models, which are based on the finite difference and split Bregman schemes.
 - ✓ The models effectively reduce the staircase (blocking) artifacts and outperforms the existing approaches for image denoising applications.
-
- ✍ Compare the results with the **splitting-convexity** method, prove the **stability** and **convergence** of this scheme.
 - ✍ Study the **segmentation** problem by the **Level set method** using the **ADMM** scheme.
 - ✍ Upgrade the results using **Machine Learning** (ML).

Thank You for Your Attention!

TG (4th and 2nd order PDEs)

Difference between TG indicators



- Profiles:

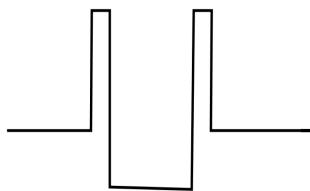
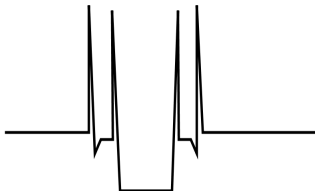
Topological gradient - 4th order PDE

Topological gradient - 2nd order PDE

Light gray

Gray

Black



Numerical computation: Model 1



How to solve, numerically, the nonlinear problem (3)?

We transform our problem to the time-dependent one:

$$\begin{cases} u_t + \nabla(\lambda \nabla u) + W'(u) = 0, & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega. \\ u(\cdot, t = 0) = f, & \text{in } \Omega. \end{cases} \quad (15)$$

We partition the the time interval $[0, T]$ into M equally-spaced sub-intervals

$I_k = [t^{k-1}, t^k]$, with $t^k = k\Delta t$ and $\Delta t = T/M$ is the time-step.

Let $u_h(x, t) = U^k(x)$ to be the fully-discrete solution of (15). The fully-discrete finite-element approximation to (15) is:

$$\begin{cases} \text{Find } U^k \in X_h \text{ such that} \\ (U^k, \phi_h) = (U^{k-1}, \phi_h) + \Delta t (\lambda \nabla U^{k-1}, \nabla \phi_h) + \Delta t (W'(U^{k-1}), \phi_h), \end{cases}$$

where X_h is the finite-element space defined as

$$X_h = \{\phi \in C(\overline{\Omega}); \phi|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_h\},$$